Groupwork 9 Solution
MATH 241 (Spring 2023)
04/18/2023

Problem 1.

(a) In the first quadrant (i.e. $x, y \geq 0$), $\boldsymbol{F}(x, y) = (x, -y)$ should face southeast $\searrow$. The only plot that satisfies this is IV.

(b) If you go along the $x = y$ line of the plot of $\boldsymbol{F}(x, y) = (y, x - y)$, we see that the y-coordinate is always 0, i.e. the vectors are horizontal (either $\leftarrow$ or $\rightarrow$), so the corresponding plot is III.

(c) In the first quadrant, $\boldsymbol{F}(x, y) = (y, y + 2)$ should face northeast $\nearrow$. The only plot that matches is I.

(d) If you go along the $y = -x$ line, the x-coordinate of $\boldsymbol{F}(x, y) = (\cos(x+y), x)$ will always be 1. Hence the arrow has to always face $\nearrow$ or $\searrow$. The y-coordinate of $\boldsymbol{F}$ is negative on the second quadrant, so the arrows face $\searrow$, and the y-coordinate is positive on the fourth quadrant, so the arrows face $\nearrow$. One can, thus, conclude that the plot is II.

Problem 2. Note that ∇ and curl always outputs a vector, so if the quantities **makes sense**, then (a), (b), (c), (d), and (f) are vector-valued. The divergence div outputs a scalar, so if the quantities **makes sense**, then (e) and (g) are scalar-valued. Both curl and div takes vector as a input, so (f) and (g) does not make sense.

(a) vector-valued

(b) vector-valued

(c) vector-valued

(d) vector-valued

(e) scalar-valued

(f) undefined

(g) undefined

Problem 3.

$$\mathrm{curl}\,\boldsymbol{F} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3 + 2xy & x^2 - 3y^2 & 0 \end{vmatrix} = (0-0)\hat{\boldsymbol{\imath}} - (0-0)\hat{\boldsymbol{\jmath}} + (2x - 2x)\hat{\boldsymbol{k}} = \boldsymbol{0}$$

Since $\boldsymbol{F}(x, y, z)$ is defined on all real space $\mathbb{R}^3$, so this vector field is conservative. If such function f exists, we have

$$\frac{\partial f}{\partial x} = 3 + 2xy \quad \text{and} \quad \frac{\partial f}{\partial y} = x^2 - 3y^2 \quad \text{and} \quad \frac{\partial f}{\partial z} = 0$$

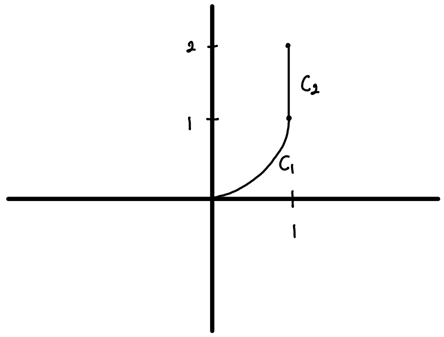
Then

$$f = \int (3 + 2xy)\, dx = 3x + x^2 y + g(y, z)$$

Consider $\frac{\partial f}{\partial y} = x^2 + \frac{\partial g}{\partial y}(y, z) = x^2 - 3y^2$. Then $\frac{\partial g}{\partial y}(y, z) = -3y^2$, so $g = \int \frac{\partial g}{\partial y}(y, z)\, dy = \int -3y^2\, dy = -y^3 + h(z)$. Therefore $f = x^2 - 3y^2 + h(z)$. As $\frac{\partial f}{\partial z} = h'(z) = 0 \Rightarrow f = 3x + x^2 y - y^3 + C$. We can take $C = 0$, so a potential function is

$$\boxed{f(x, y, z) = 3x + x^2 y - y^3}$$

Problem 4.



The parametrization of C_1 is given by $\boldsymbol{r}_1(t) = (t, t^2)$ for $0 \le t \le 1$ and the parametrization of C_2 is given by $\boldsymbol{r}_2(t) = (1, t)$ for $1 \le t \le 2$.

$$\begin{aligned}\int_C \rho(x,y)\,ds &= \int_{C_1} \rho(x,y)\,ds + \int_{C_2} \rho(x,y)\,ds \\ &= \int_0^1 (2t+1)\sqrt{1+4t^2}\,dt + \int_1^2 3\,dt\end{aligned}$$

Since we have

$$\int_0^1 (2t+1)\sqrt{1+4t^2}\,dt = \int_0^1 2t\sqrt{1+4t^2}\,dt + \int_0^1 \sqrt{1+4t^2}\,dt$$

we compute individually. Using u-substitution $u = 1+4t^2$, we have

$$\int_0^1 2t\sqrt{1+4t^2}\,dt = \frac{1}{4}\int_0^1 \sqrt{u}\,dt = \frac{1}{4}\left(\frac{2}{3}u^{\frac{2}{3}}\Big|_0^1\right) = \frac{1}{6}$$

as $du = 8t\,dt \Rightarrow \frac{1}{4}du = 2t\,dt$.

Warning: The following integral is more than difficult than intended. You will not need to know how to integrate the following integral.

$$\int_0^1 \sqrt{1+4t^2}\,dt = (*)$$

Let $2t = \tan\theta$. Then $1+4t^2 = 1+\tan^2\theta = \sec^2\theta$. Then $2dt = \sec^2\theta\,d\theta$. Thus the integral becomes

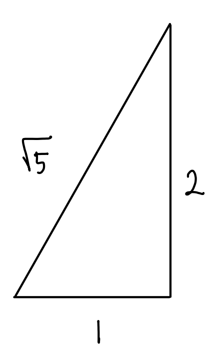
$$(*) = \frac{1}{2}\int_{t=0}^{t=1} \sec^2\theta\sqrt{\sec^2\theta}\,d\theta = \frac{1}{2}\int_{t=0}^{t=1} \sec^3\theta\,d\theta$$

Here we use (see page 518 of the textbook)

$$\int \sec^3\theta\,d\theta = \underbrace{\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta|}_{=H(\theta)} + C$$

As $\theta = \tan^{-1}(2t)$, we have

$$\int_{t=0}^{t=1} \sec^3\theta\,d\theta = \int_{\tan^{-1}(0)}^{\tan^{-1}(2)} \sec^3\theta\,d\theta = H(\tan^{-1}(2)) - H(\tan^{-1}(0))$$



So we have $\sec(\tan^{-1}(2)) = \sqrt{5}$. Therefore

$$\begin{aligned} H(\tan^{-1}(2)) &= \frac{1}{2}\sec(\tan^{-1}(2))\tan(\tan^{-1}(2)) + \frac{1}{2}\ln|\sec(\tan^{-1}(2)) + \tan(\tan^{-1}(2))| \\ &= \sqrt{5} + \frac{1}{2}\ln(\sqrt{5}+2) \end{aligned}$$

And since $\tan^{-1}(0) = 0$, we have

$$H(\tan^{-1}(0)) = \frac{1}{2}\sec(0)\tan(0) + \frac{1}{2}\ln|\sec(0)+\tan(0)| = 0$$

To summarize,

$$\int_0^1 \sqrt{1+4t^2}\,dt = \frac{1}{2}\left(\sqrt{5} + \frac{1}{2}\ln(\sqrt{5}+2)\right)$$

Finally, to conclude,

$$\begin{aligned} \int_C \rho(x,y)\,ds &= \int_{C_1} \rho(x,y)\,ds + \int_{C_2} \rho(x,y)\,ds \\ &= \int_0^1 2t\sqrt{1+4t^2}\,dt + \int_0^1 \sqrt{1+4t^2}\,dt + \int_1^2 3\,dt \\ &= \boxed{\frac{1}{6} + \frac{1}{2}\left(\sqrt{5} + \frac{1}{2}\ln(\sqrt{5}+2)\right) + 3} \end{aligned}$$

Problem 5. The quarter circle in the first quadrant with raidues 2 starting from $(2,0)$ is given by the paramterization $\boldsymbol{r}_1(t) = (2\cos t, 2\sin t)$ for $0 \le t \le \frac{\pi}{2}$. Therefore the work done is

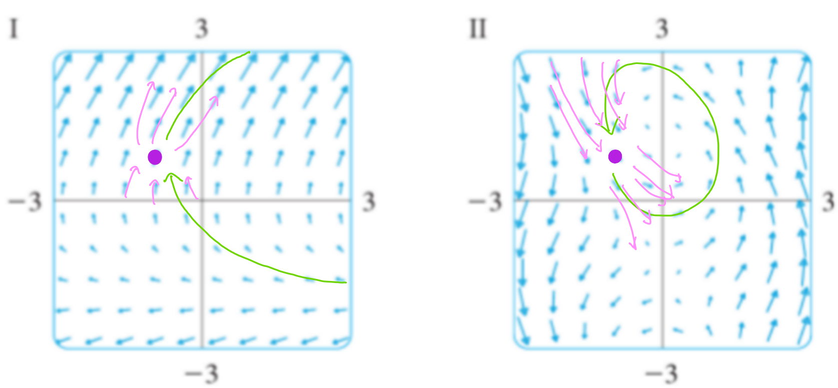
$$\begin{aligned} \int_C \boldsymbol{F}\cdot d\boldsymbol{r}_1 &= \int_0^{\frac{\pi}{2}} (4\cos^2 t, -4\cos t\sin t)\cdot(-2\sin t, 2\cos t)\,dt \\ &= \int_0^{\frac{\pi}{2}} -8\cos^2 t\sin t - 8\cos^2 t\sin t\,dt \\ &= -16\int_0^{\frac{\pi}{2}} \cos^2 t\sin t\,dt \\ &= 16\int_1^0 u^2\,du = \boxed{-\frac{16}{3}} \end{aligned}$$

If $\boldsymbol{r}_2(t) = (2\cos t, 2\sin t)$ for $0 \le t \le 2\pi$, then

$$\int_C \boldsymbol{F}\cdot d\boldsymbol{r}_2 = -16\int_0^{2\pi} \cos^2 t\sin t\,dt = 16\int_1^1 u^2\,du = \boxed{0}$$

Problem 6.

Around the point $(-2,2)$ of plot I, the flow is more outward than inward, so $\operatorname{div} \boldsymbol{F}$ is expected to be positive. Also, the vectors rotate clockwise, so we expect $\operatorname{curl} \boldsymbol{F}$ to be in the direction $-\hat{\boldsymbol{k}}$. For plot II, the same amount flows into and out of the point $(-2,2)$, so the divergence $\operatorname{div} \mathbb{F} = 0$. The vectors rotate counterclockwise, so we expect $\operatorname{curl} \boldsymbol{F}$ to be in the direction $\hat{\boldsymbol{k}}$.



We use $\boldsymbol{F}_1(x, y) = (y, y+2)$ for I. Then $\operatorname{curl} \boldsymbol{F}_1 = (0-1)\hat{\boldsymbol{k}} = -\hat{\boldsymbol{k}}$, and $\operatorname{div} \boldsymbol{F}_1 = 0 + 1 = 1$. Next, we use $\boldsymbol{F}_2(x, y) = (\cos(x+y), x)$ for II. Then $\operatorname{curl} \boldsymbol{F}_2(x, y) = (1 - \sin(x+y))\hat{\boldsymbol{k}}$, so $\operatorname{curl} \boldsymbol{F}_2(-2, 2) = (1 - \sin(0))\hat{\boldsymbol{k}} = \hat{\boldsymbol{k}}$. Also, $\operatorname{div} \boldsymbol{F}_2(x, y) = -\sin(x+y) = 0$.