Groupwork 10 Solution
MATH 241 (Spring 2023)
04/25/2023

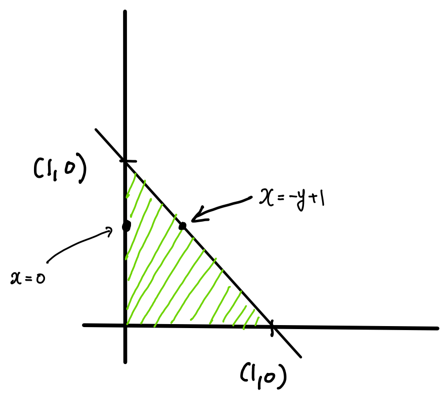
Problem 1. The area R bounded by the circle $x^2 + y^2 = 4$ is given by

$$R = \{(r, \theta) \mid 0 \le r \le 2 \text{ and } 0 \le \theta \le 2\pi\}$$

Then by Green's Theorem we have

$$\begin{aligned}
\int_C x^2 y\, dx - xy^2\, dy &= \iint_R -y^2 - x^2\, dA \\
&= \int_0^{2\pi} \int_0^2 -r^2 \cdot r\, dr\, d\theta \\
&= \int_0^{2\pi} \int_0^2 -r^3\, dr\, d\theta \\
&= \int_0^{2\pi} -\frac{r^4}{4}\bigg|_0^2\, d\theta \\
&= \int_0^{2\pi} -4\, d\theta = \boxed{-8\pi}
\end{aligned}$$

Problem 2.



$$\begin{aligned}
\int_C x^4\, dx + xy\, dy &= \iint_R y\, dA \\
&= \int_0^1 \int_0^{-y+1} y\, dx\, dy \\
&= \int_0^1 y(-y+1)\, dy \\
&= \int_0^1 y - y^2\, dy \\
&= \frac{y^2}{2} - \frac{y^3}{3}\bigg|_0^1 = \boxed{\frac{1}{6}}
\end{aligned}$$

Problem 3. We solve (a) and (b) together. We show that $\boldsymbol{F}$ is conservative. In other words, there exists a potential function f of $\boldsymbol{F}$ such that $\boldsymbol{F} = \mathrm{grad}\, f$. If such f exists, $f_x = 2xy$ and $f_y = x^2$. Then

$$f = \int f_x\, dx = \int 2xy\, dx = x^2 y + h(y)$$

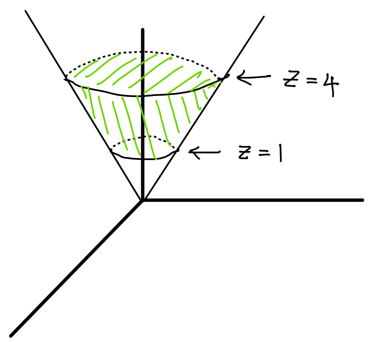
Hence $f_y = x^2 + h'(y)$, so we have $x^2 = x^2 + h'(y) \Rightarrow h'(y) = 0$. Therefore, y is a constant. Since we can choose any constant, we set the constant to be 0. So $f = x^2 y$. This shows that $\boldsymbol{F}$ is conservative.

Then by the fundamental theorem of line integral

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_C \text{grad } f \cdot d\boldsymbol{r} = f(3,2) - f(1,2) = 18 - 2 = \boxed{16}$$

which is same for all curves C that have the same starting point $(1,2)$ and end point $(3,2)$.

Problem 4. We use *polar coordinates*, i.e. let $x = r\cos\theta$ and $y = r\sin\theta$. Then $z = \sqrt{x^2+y^2} = \sqrt{r^2} = r$.



Then a parametrization of the thin cone is

$$\boldsymbol{s}(r,\theta) = (r\cos\theta, r\sin\theta, r)$$

for $1 \leq r \leq 4$ and $0 \leq \theta \leq 2\pi$. For later use, we compute $||\boldsymbol{s}_r \times \boldsymbol{s}_\theta||$.

$$||\boldsymbol{s}_r \times \boldsymbol{s}_\theta|| = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = ||(-r\cos\theta, -r\sin\theta, r)|| = \sqrt{2r^2} = \sqrt{2}r$$

Then the total mass is

$$\iint_\Sigma 10 - z\ dS = \int_0^{2\pi}\int_1^4 (10-r)\sqrt{2}r\ dr\ d\theta = \sqrt{2}\int_0^{2\pi}\int_1^4 10r - r^2\ dr\ d\theta$$

Since

$$\int_1^4 10r - r^2\ dr = 5r^2 - \frac{r^3}{3}\bigg|_1^4 = \left(\left(80 - \frac{64}{3}\right) - \left(5 - \frac{1}{3}\right)\right) = 75 - \frac{63}{3} = 75 - 21 = 54$$

Therefore the total mass is

$$\sqrt{2}\int_0^{2\pi}\left(\int_1^4 10r - r^2\ dr\right)\ d\theta = \sqrt{2}\int_0^{2\pi} 54\ d\theta = \boxed{108\sqrt{2}\pi}$$