Groupwork 11 Solution
MATH 241 (Spring 2023)
05/02/2023

Problem 1. We solve this by doing two variable changes. Let

$$x = au, \qquad y = bv, \qquad z = \sqrt{c}w$$

Then the ellipsoid transforms to a sphere

$$1 = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c} = \frac{a^2u^2}{a^2} + \frac{b^2v^2}{b^2} + \frac{cw^2}{c} = u^2 + v^2 + w^2$$

in the (u, v, w)-coordinate system. Then the Jacobian is

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & \sqrt{c} \end{vmatrix} = ab\sqrt{c}$$

and the integral becomes

$$\iiint_V dV = ab\sqrt{c} \iiint_E dE$$

where E is the sphere of radius 1 in the (u, v, w)-coordinate system. As $\iiint_E dE$ is the volume of the sphere, $\iiint_E dE = \frac{4}{3}\pi$. Another way to see this is do another change of variables

$$u = \rho \sin\phi \cos\theta, \qquad v = \rho \sin\phi \sin\theta, \qquad , w = \rho \cos\phi$$

Then

$$\begin{aligned} \iiint_E dE &= \int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \\ &= \int_0^{2\pi} \int_0^{\pi} \frac{\rho^3}{3} \sin\phi \Big|_0^1 \, d\phi \, d\theta \\ &= \frac{1}{3} \int_0^{2\pi} \int_0^{\pi} \sin\phi \, d\phi \, d\theta \\ &= \frac{1}{3} \int_0^{2\pi} (-\cos\phi)\Big|_0^{\pi} \, d\theta \\ &= \frac{2}{3} \int_0^{2\pi} d\theta = \boxed{\frac{4\pi}{3}} \end{aligned}$$

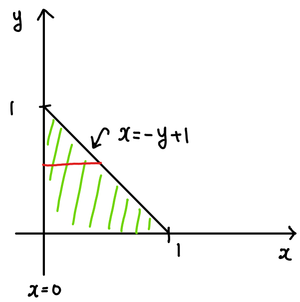
Problem 2. We have

$$f_x = yz, \qquad f_y = xz, \qquad f_z = xy + 2z$$

Then $f = \int f_x \, dx = xyz + g(y, z)$. Taking partial y, $f_y = xz + g_y(y, z) = xz \Rightarrow g_y(y, z) = 0 \Rightarrow g(y, z) = h(z)$. Hence we have $f = xyz + h(z)$. Finally, $f_z = xy + h'(z) = xy + 2z$. Then $h'(z) = 2z \Rightarrow h(z) = z^2 + C$. By setting $C = 0$, we have a potential function $f = xyz + z^2$. Then by the fundamental theorem of line integrals

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = f(4, 6, 3) - f(1, 0, -2) = 72 + 9 - (4 - 0) = \boxed{77}$$

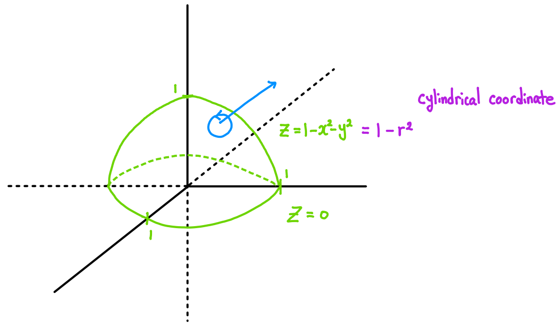
Problem 3.



Then by Green's theorem

$$
\begin{aligned}
\int_C x^4\,dx + xy\,dy &= \iint_R \frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(x^4)\,dA \\
&= \iint_R y\,dA \\
&= \int_0^1 \int_0^{-y+1} y\,dx\,dy \\
&= \int_0^1 y(-y+1)\,dy \\
&= \int_0^1 y - y^2\,dy \\
&= \frac{y^2}{2} - \frac{y^3}{3}\bigg|_0^1 = \frac{1}{2} - \frac{1}{3} = \boxed{\frac{1}{6}}
\end{aligned}
$$

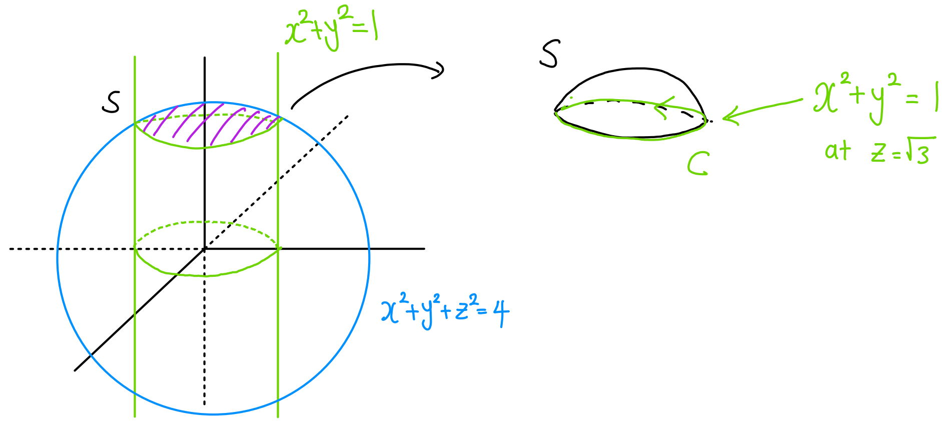
Problem 4.



Then by divergence theorem

$$
\begin{aligned}
\iint_C \boldsymbol{F} \cdot \boldsymbol{n}\, dS &= \iiint_D \operatorname{div} \boldsymbol{F}\, dV \\
&= \iiint_D dV \\
&= \int_0^{2\pi} \int_0^1 \int_0^{1-r^2} r\, dz\, dr\, d\theta \\
&= \int_0^{2\pi} \int_0^1 r(1-r^2)\, dr\, d\theta \\
&= \int_0^{2\pi} \int_0^1 r - r^3\, dr\, d\theta \\
&= \int_0^{2\pi} \frac{r^2}{2} - \frac{r^4}{4}\Big|_0^1\, d\theta \\
&= \int_0^{2\pi} \left(\frac{1}{2} - \frac{1}{4}\right) d\theta = \frac{1}{4} \cdot 2\pi = \boxed{\frac{\pi}{2}}
\end{aligned}
$$

Problem 5.



Then the parametrization of the boundary of S is given by the intersection of the sphere and the cylinder. Since it is also part of the cylinder $x^2 + y^2 = 1$, it will be a circle of radius 1 at $z = \sqrt{3}$. To find z, we substitute $x^2 + y^2 = 1$ to the equation of the sphere. Then we have $1 + z^2 = 4 \Rightarrow z^2 = 3 \Rightarrow z = \sqrt{3}$. Hence the parametrization of the boundary C of S and its derivative are given by

$$\boldsymbol{r}(t) = (\cos t, \sin t, \sqrt{3}) \quad \text{and} \quad \boldsymbol{r}'(t) = (-\sin t, \cos t, 0)$$

for $0 \leq t \leq 2\pi$. Then Stokes's theorem shows that

$$\iint_S \operatorname{curl} \boldsymbol{F} \cdot \boldsymbol{n}\, dS = \int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_0^{2\pi} \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t)\, dt = \int_0^{2\pi} 0\, dt = \boxed{0}$$

To see this,

$$\boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t) = (\sqrt{3}\cos t, \sqrt{3}\sin t, \sin t \cos t) \cdot (-\sin t, \cos t, 0) = -\sqrt{3}\sin t \cos t + \sqrt{3}\sin t \cos t + 0 = 0$$