① (a) $\displaystyle\int_0^{\frac{\pi}{2}}\int_0^{\sin\theta} r\cos\theta \, dr\, d\theta$

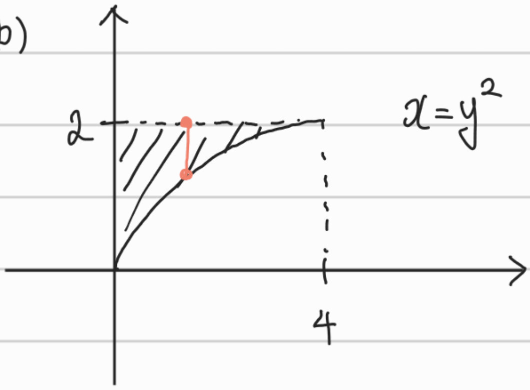
$$= \int_0^{\frac{\pi}{2}} \cos\theta \left.\frac{r^2}{2}\right|_0^{\sin\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^2\theta\cos\theta \, d\theta \qquad \begin{array}{l} u=\sin\theta \\ du=\cos\theta\, d\theta \end{array}$$

$$= \frac{1}{2}\int_0^1 u^2\, du$$

$$= \frac{1}{2}\cdot\left.\frac{u^3}{3}\right|_0^1 = \frac{1}{6}$$

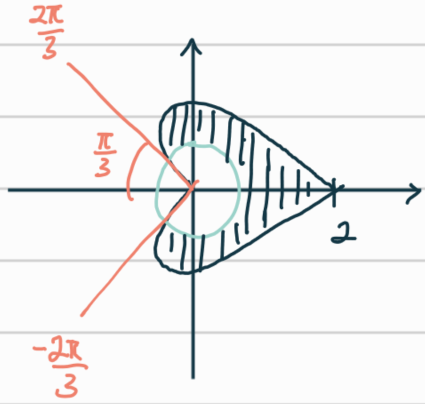
(b)



$x=y^2 \Rightarrow y=\sqrt{x}$

$$\int_0^4\int_{\sqrt{x}}^2 xy \, dx\, dy$$

② (a)



$r = 1+\cos\theta$

$r = \frac{1}{2}$

$$\int_{-\frac{2\pi}{3}}^{\frac{2\pi}{3}} \int_{\frac{1}{2}}^{1+\cos\theta} r^2 \sin\theta \quad dr\, d\theta$$

(b)

$y = -x-2$

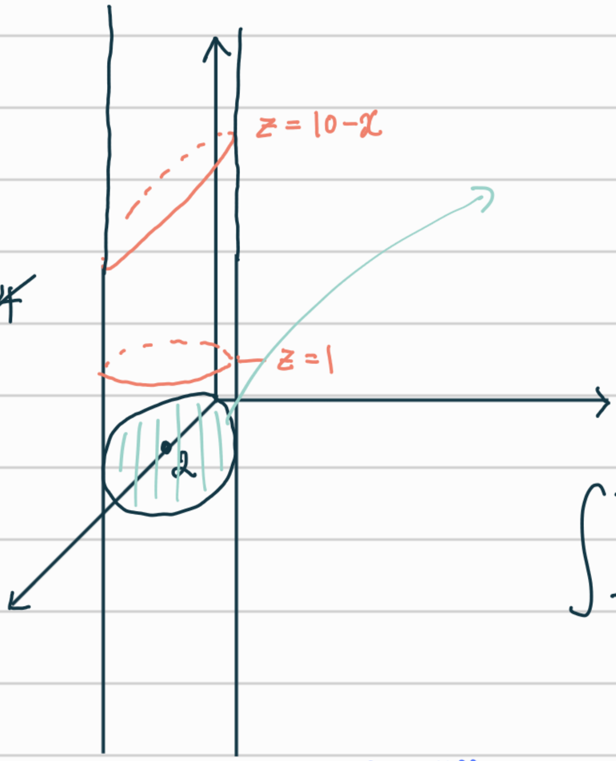
$\Rightarrow r\sin\theta = -r\cos\theta - 2$

$\Rightarrow r(\sin\theta - \cos\theta) = -2$

$\Rightarrow r = \dfrac{-2}{\sin\theta - \cos\theta} = \dfrac{2}{\cos\theta - \sin\theta}$

$$\int_{\pi}^{\frac{3\pi}{2}} \int_{0}^{\frac{2}{\cos\theta - \sin\theta}} r^2 \sin\theta \quad dr\, d\theta$$

③



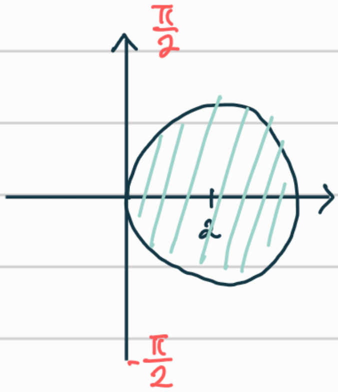
$(x-2)^2+y^2=4$

$x^2-4x+\cancel{4}+y^2=\cancel{4}$

$x^2+y^2=4x$

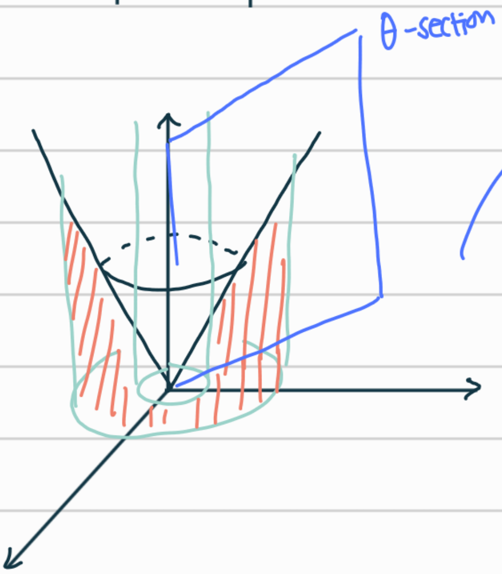
$r^2=4r\cos\theta$

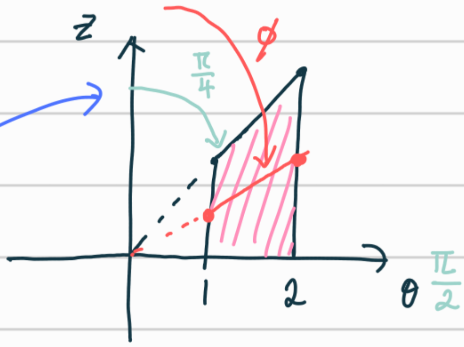
$r=4\cos\theta$



$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{4\cos\theta}\int_1^{10-r\cos\theta} dz\,dr\,d\theta$$

④





Convert the equation of the cone to get

$$z=\sqrt{x^2+y^2}$$

$$\Rightarrow \rho\cos\phi = \sqrt{\rho^2\sin^2\phi} = \rho\sin\phi$$

$$\Rightarrow \cos\phi = \sin\phi$$

Hence $\phi = \frac{\pi}{4}$

Using this we get that ϕ range is

$$\frac{\pi}{4}\le\phi\le\frac{\pi}{2}$$

For ρ, we have

$$x^2+y^2=1 \Rightarrow \rho^2\sin^2\phi=1 \Rightarrow \rho\sin\phi=1 \Rightarrow \rho=\frac{1}{\sin\phi}$$

$$x^2+y^2=4 \Rightarrow \rho^2\sin^2\phi=4 \Rightarrow \rho\sin\phi=2 \Rightarrow \rho=\frac{2}{\sin\phi}$$

all together, we have

$$\int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_{\frac{1}{\sin\phi}}^{\frac{2}{\sin\phi}}(\rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

⑤

$$\begin{aligned} y &= \tfrac{1}{2}x + 1 \\ y &= \tfrac{1}{2}x + 3 \\ y &= 5 - x \\ y &= 2 - x \end{aligned} \quad \Rightarrow \quad \begin{aligned} u &= -\tfrac{1}{2}x + y = 1 \\ u &= -\tfrac{1}{2}x + y = 3 \\ v &= x + y = 5 \\ v &= x + y = 2 \end{aligned}$$

Then $\dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} -\frac{1}{2} & 1 \\ 1 & 1 \end{vmatrix} = -\frac{1}{2} - 1 = -\frac{3}{2}$

$$\Rightarrow \quad \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \left| -\frac{2}{3} \right| = \frac{2}{3}$$

$v - u = \frac{3}{2}x \Rightarrow x = \frac{2}{3}(v-u)$

$2u + v = (-x+2y) + x + y = 3y \Rightarrow y = \frac{1}{3}(2u+v)$

$$\iint_R xy \, dA = \int_2^5 \int_1^3 \frac{2}{3}(v-u) \cdot \frac{1}{3}(2u+v) \cdot \frac{2}{3} \, du \, dv$$

$$= \int_2^5 \int_1^3 \frac{4}{27}(v-u)(2u+v) \, du \, dv.$$