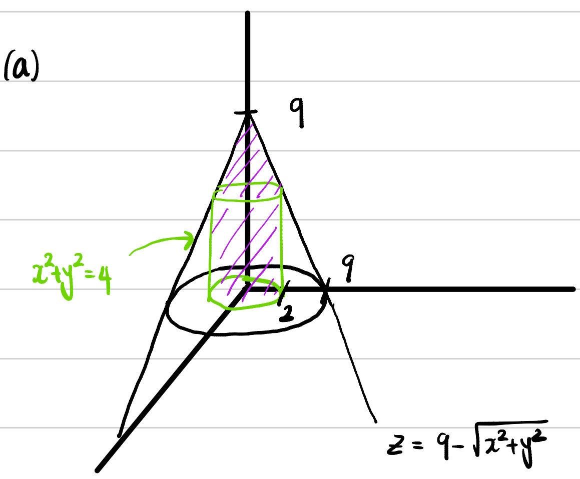
Justin Fall 2015 Exam 3 Solution

1) (a)



when $z=0$, we get $0 = 9-\sqrt{x^2+y^2}$

$\Rightarrow 9^2 = x^2+y^2$, which is the circle of radius 9

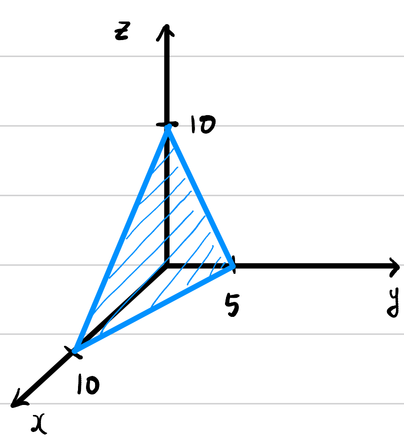
But the projection of $x^2+y^2=4$ on the xy-plane is the circle of radius 2, so we get

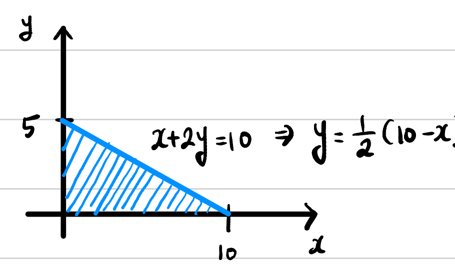
$\{(r,\theta) \mid 0 \le r \le 2 \text{ and } 0 \le \theta \le 2\pi\}$

Hence the iterated integral is

$$\int_0^{2\pi}\int_0^{2}\int_0^{9-\sqrt{x^2+y^2}} r \, dz\, dr\, d\theta$$

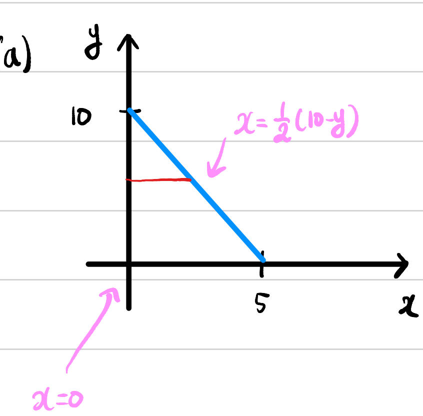
(b)





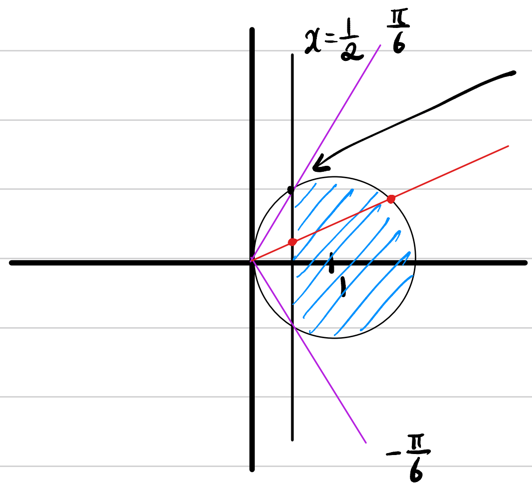
$$\int_0^{10}\int_0^{\frac{1}{2}(10-x)}\int_0^{10-x-2y} xy \, dz\, dy\, dz$$

2) (a)



$$\int_0^{10}\int_0^{\frac{1}{2}(10-y)} y \, dx\, dy$$

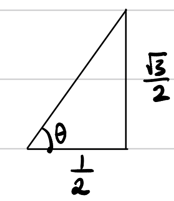
(b)



By substituting $x=\frac{1}{2}$,

$$\left(\frac{1}{2}\right)^2 + y^2 = 1$$

$$\Rightarrow y = \frac{\sqrt{3}}{2}$$



$$\Rightarrow \tan\theta = \sqrt{3} \Rightarrow \theta = \frac{\pi}{6}$$

This shows that $-\frac{\pi}{6} \leq \theta \leq \frac{\pi}{6}$

$$x = \frac{1}{2} \Rightarrow r\cos\theta = \frac{1}{2} \Rightarrow r = \frac{1}{2\cos\theta}$$

$$(x-1)^2 + y^2 = 1$$

$$\Rightarrow x^2 - 2x + \cancel{1} + y^2 = \cancel{1}$$

$$\Rightarrow x^2 + y^2 = 2x$$

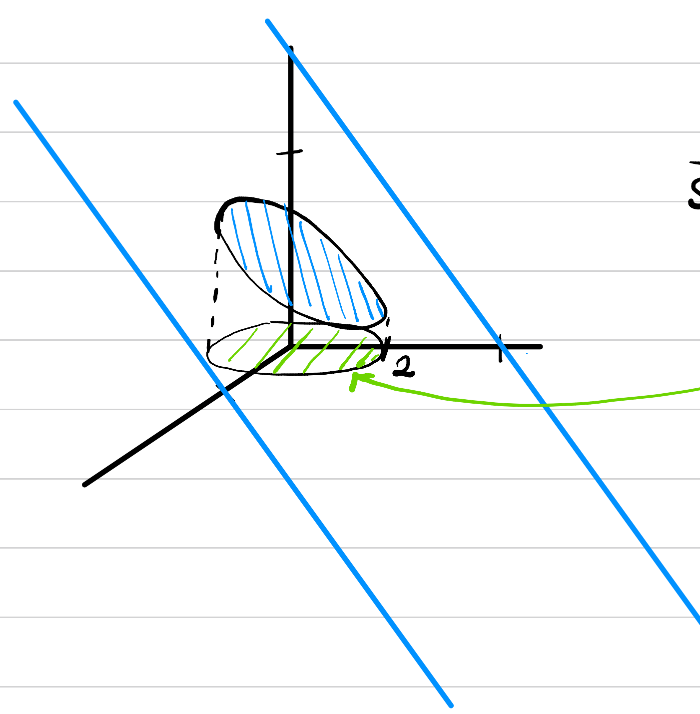
$$\Rightarrow r^2 = 2r\cos\theta$$

$$\Rightarrow r = 2\cos\theta$$

$$\rightarrow \frac{1}{2\cos\theta} \leq r \leq 2\cos\theta$$

$$\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \int_{\frac{1}{2\cos\theta}}^{2\cos\theta} (r\cos\theta - r\sin\theta)\, r\, dr\, d\theta$$

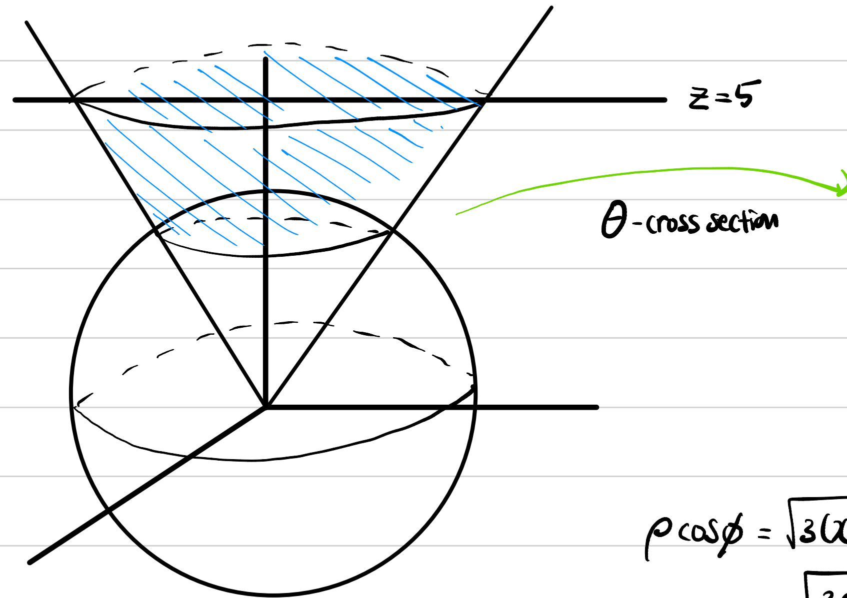
3) (a)

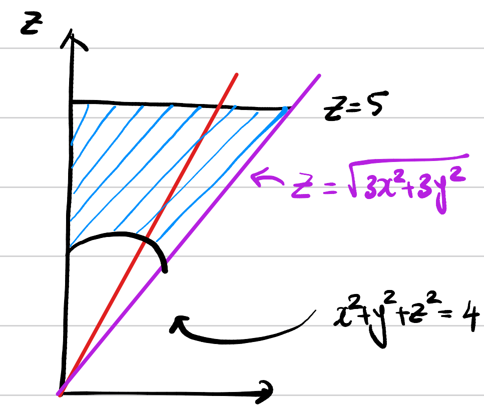


$$\vec{S}(r,\theta) = \underbrace{r\cos(\theta)\hat{i} + r\sin(\theta)\hat{j}} + (10 - 3r\cos\theta - r\sin\theta)\hat{k}$$

$$0 \leq \theta \leq 2\pi$$
$$0 \leq r \leq 2$$

(b)



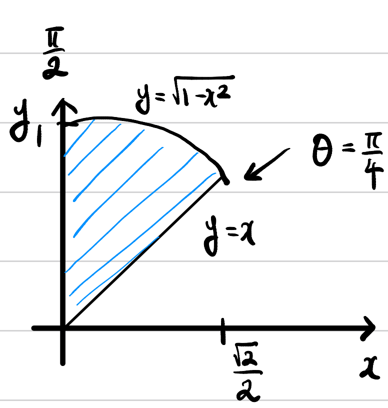


$$\rho\cos\phi = \sqrt{3(x^2+y^2)}$$
$$= \sqrt{3\rho^2\sin^2\phi}$$
$$= \sqrt{3}\rho\sin\phi$$
$$\Rightarrow \tan\phi = \frac{1}{\sqrt{3}} \Rightarrow \phi = \frac{\pi}{6}$$

$$z=5 \rightsquigarrow \rho\cos\phi = 5 \rightsquigarrow \rho = \frac{5}{\cos\phi}$$

$$\int_0^{2\pi}\int_0^{\frac{\pi}{6}}\int_2^{\frac{5}{\cos\phi}} (\rho\cos\phi)\rho^2\sin\phi \, d\rho\, d\phi\, d\theta$$

4)



$$0 \leq r \leq 1$$
$$\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$$

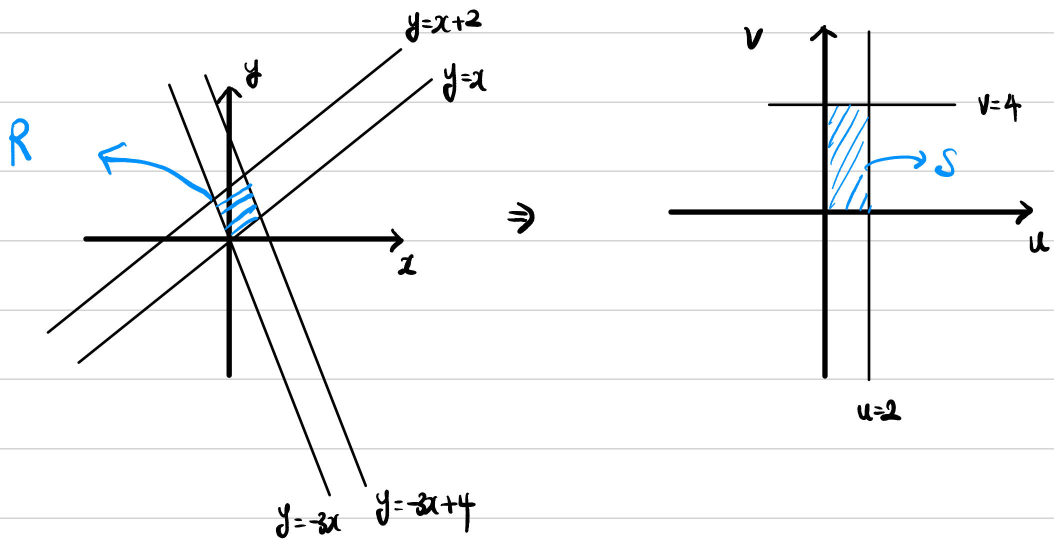
$$\Rightarrow \int_0^{\frac{\sqrt{2}}{2}}\int_x^{\sqrt{1-x^2}} e^{(x^2+y^2)}\, dy\, dx$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\int_0^1 e^{r^2} r\, dr\, d\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^1 e^u\, du\, d\theta = \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (e-1)\, d\theta = \frac{\pi}{8}(e-1).$$

5. $\left.\begin{array}{l} y-x=0 \\ y-x=2 \\ 3x+y=0 \\ 3x+y=4 \end{array}\right\} \Rightarrow \begin{array}{l} u=0 \\ u=2 \\ v=0 \\ v=4 \end{array}$ if $u=y-x$ and $v=3x+y$

$v-u = (3x+y)-(y-x) = 4x \Rightarrow \frac{1}{4}(v-u)$



$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} -1 & 1 \\ 3 & 1 \end{vmatrix} = -4 \Rightarrow \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = 1/4$$

$$\int_0^2 \int_0^4 \frac{1}{4}(v-u) \cdot \frac{1}{4}\, dv\, du = \int_0^2 \int_0^4 \frac{1}{16}(v-u)\, dv\, du$$