1. (a) Find the mass of the wire segment C (evaluate!) parametrized by $\mathbf{r}(t) = t\,\mathbf{i} + (3t+1)\,\mathbf{j}$ for $0 \le t \le 4$ if the density is given by $f(x, y) = xy$.

(b) Evaluate $\int_C 2y\ dx + (2x+z)\ dy + y\ dz$ where C is the curve parametrized by $\mathbf{r}(t) = (t^2+1)\,\mathbf{i} + \sqrt{t}\,\mathbf{j} - 4t\,\mathbf{k}$ for $1 \le t \le 4$.

(a) $\vec{r}\,'(t) = (1,3)$

$\|\vec{r}\,'(t)\| = \sqrt{1+9} = \sqrt{10}$

$$\int_C f\,d\vec{r} = \int_0^4 t(3t+1)\cdot\sqrt{10}\,dt = \sqrt{10}\int_0^4 3t^2+t\,dt = \sqrt{10}\ \left. t^3+\frac{t^2}{2}\right|_0^4$$
$$= \sqrt{10}\,(64+8)$$
$$= 72\sqrt{10}$$

(b) We check if the vector field $\vec{F}(x,y,z) = (2y,\ 2x+z,\ y)$ is conservative.

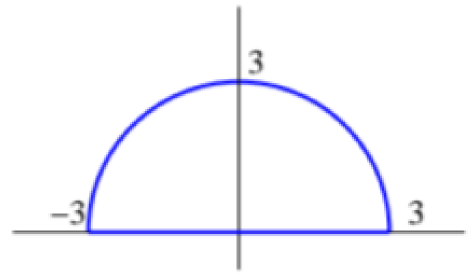
$$\left.\begin{array}{lll} f_x = 2y & \Rightarrow & f = 2xy + g_1(y,z) \\ f_y = 2x+z & \Rightarrow & f = 2xy + yz + g_2(x,z) \\ f_z = y & \Rightarrow & f = yz + g_3(x,y) \end{array}\right\} \Rightarrow\ f = 2xy + yz \text{ works.}$$

$\vec{r}(1) = (2,1,-4)$

$\vec{r}(4) = (17,2,-16)$, so

$$\int_C \vec{F}\cdot d\vec{r} = f(17,2,-16) - f(2,1,-4)$$
$$= 2\cdot 17\cdot 2 + 2\cdot(-16) - (4+(-4))$$
$$= 4\cdot 17 - 2\cdot 16$$
$$= 68 - 32 = \boxed{36}$$

2. Evaluate $\int_C 2y^2\ dx + 3y\ dy$ where C is the semicircle shown with clockwise orientation: You must evaluate this integral!



We use Green's theorem

clockwise orientation

$$\int_C 2y^2\,dx + 3y\,dy = -\iint_R -4y\ dA$$

$$= \iint_R 4y\ dA$$

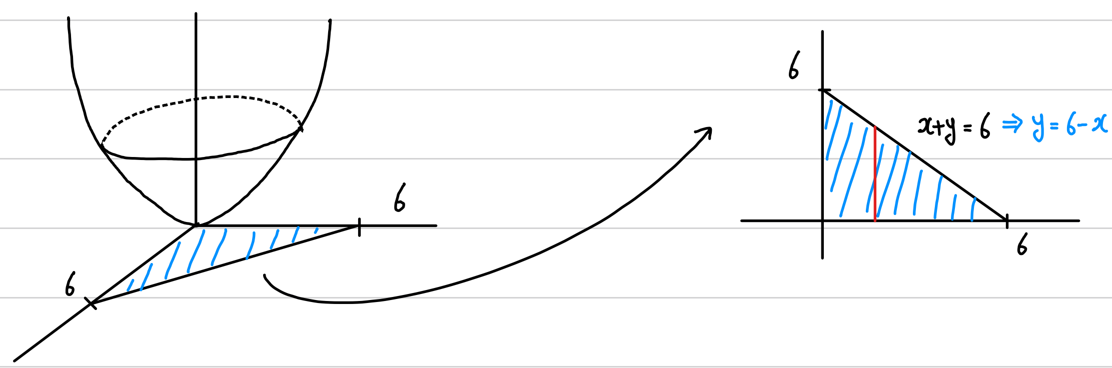
$$= \int_0^{\pi}\int_0^3 4r\sin\theta \cdot r\ dr\,d\theta$$

$$= \int_0^{\pi}\int_0^3 4r^2\sin\theta\ dr\,d\theta = \int_0^{\pi} \tfrac{4}{3}r^3\sin\theta \Big|_0^3\ d\theta$$

$$= \int_0^{\pi} 36\sin\theta\ d\theta$$

$$= -36\cos\theta\Big|_0^{\pi} = 72$$

3. Suppose Σ is the surface consisting of the portion of the paraboloid $z = x^2 + y^2$ above the filled-in triangle in the xy-plane with corners $(0,0,0)$, $(0,6,0)$ and $(6,0,0)$. If the density is given by $f(x,y,z) = yz$ find an iterated double integral which gives the total mass of Σ but do not evaluate.



Since Σ is part of the paraboloid $z=x^2+y^2$, we use the parametrization

$$\vec{r}(x,y) = (x, y, x^2+y^2)$$

$$0 \leq x \leq 6 \qquad 0 \leq y \leq 6-x$$

Then

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 2x \\ 0 & 1 & 2y \end{vmatrix} = (-2x, -2y, 1)$$

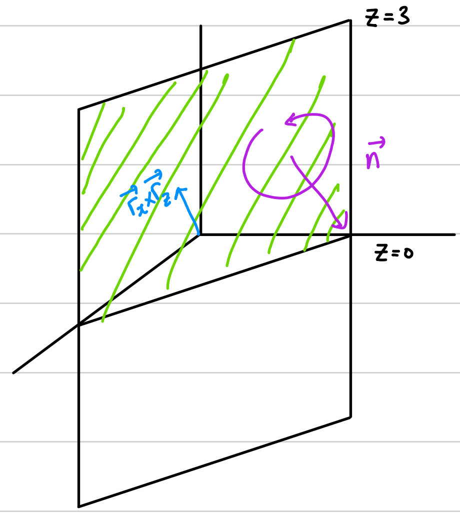
$$\|\vec{r}_x \times \vec{r}_y\| = \sqrt{4x^2+4y^2+1}$$

$$\iint_{\Sigma} f\, dS = \int_0^6 \int_0^{6-x} y(x^2+y^2)\sqrt{4x^2+4y^2+1}\; dy\, dx$$

4. Let Σ be the portion of the plane $x + y = 2$ in the first octant between $z = 0$ and $z = 3$. Let C be its edge, oriented counterclockwise when looking in towards the octant. Apply Stokes' Theorem to the line integral

$$\int_C (x\,\mathbf{i} + xy\,\mathbf{j} + xyz\,\mathbf{k}) \cdot d\mathbf{r}$$

Parametrize the resulting surface and proceed until you have an iterated double integral but do not evaluate.



Then the plane is parametrized by

$\vec{r}(x,z) = (x, 6-x, z)$

$0 \leq x \leq 6$ and $0 \leq z \leq 3$

$$\vec{r}_x \times \vec{r}_z = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = (-1, -1, 0)$$

opposite direction of $\vec{n}$

$\vec{F}(x,y,z) = (x, xy, xyz)$

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & xy & xyz \end{vmatrix} = (xz - 0)\hat{i} - (yz - 0)\hat{j} + (y - 0)\hat{k} = (xz, -yz, y)$$

Then $\displaystyle \int_C \vec{F} \cdot d\vec{r} = \iint_\Sigma (\text{curl } \vec{F}) \cdot \vec{n}\, dS$

opposite direction

$$= -\int_0^3 \int_0^6 (xz, -(6-x)z, 6-x) \cdot (-1,-1,0)\, dx\, dz$$

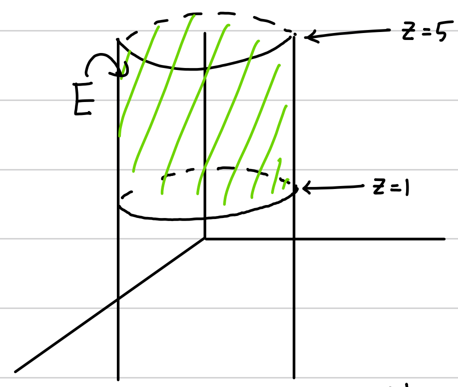
$$= \int_0^3 \int_0^6 (xz, -(6-x)z, 6-x) \cdot (1,1,0)\, dx\, dz$$

$$= \int_0^3 \int_0^6 xz - 6z + xz \; dx\, dz$$

$$= \int_0^3 \int_0^6 2xz - 6z \; dx\, dz$$

5. Suppose Σ is the portion of the cylinder $x^2 + y^2 = 9$ between $z = 1$ and $z = 5$ along with the disks which seal it off at each end. Assume Σ is oriented inwards. Use the Divergence Theorem to evaluate the following integral:

$$\iint_{\Sigma} (x\,\mathbf{i} + 2y\,\mathbf{j} + z^2\,\mathbf{k}) \cdot \mathbf{n}\; dS$$



$\vec{F}(x,y,z) = (x, 2y, z^2)$

$\operatorname{div} \vec{F} = 1 + 2 + 2z = 3 + 2z$

Polar coordinate

Then $\iint_{\Sigma} \vec{F} \cdot \vec{n}\, dS = \iiint_E 3 + 2z\, dV \overset{\downarrow}{=} - \int_0^{2\pi} \int_0^3 \int_1^5 (3+2z)\, r\, dz\, dr\, d\theta$

as Σ is oriented inwards

$$= - \int_0^{2\pi} \int_0^3 \int_1^5 3r + 2rz\, dz\, dr\, d\theta$$

$$= - \int_0^{2\pi} \int_0^3 3rz + rz^2 \Big|_1^5\, dr\, d\theta$$

$$= - \int_0^{2\pi} \int_0^3 (15r + 25r) - (3r + r)\, dr\, d\theta$$

$$= - \int_0^{2\pi} \int_0^3 36r\, dr\, d\theta$$

$$= - \int_0^{2\pi} 18r^2 \Big|_0^3\, d\theta$$

$$= - \int_0^{2\pi} 162\, d\theta = -324\pi$$