1. (a) Evaluate $\int_C 8y\, ds$ where C is parametrized by $\mathbf{r}(t) = t^2\,\mathbf{i} + t\,\mathbf{j}$ for $0 \le t \le 3$.

(b) Use the Fundamental Theorem of Line Integrals to evaluate $\int_C \left(1\,\mathbf{i} + \frac{1}{z}\,\mathbf{j} - \frac{y}{z^2}\,\mathbf{k}\right) \cdot d\mathbf{r}$ where C is parametrized by $\mathbf{r}(t) = t^2\,\mathbf{i} + t\sin(\pi t)\,\mathbf{j} + \frac{1}{t}\,\mathbf{k}$ for $\frac{1}{2} \le t \le 1$.

(a) $\vec{r}\,'(t) = (2t, 1)$

$\|\vec{r}\,'(t)\| = \sqrt{4t^2+1}$

$$\int_C 8y\, ds = \int_0^3 8t\sqrt{4t^2+1}\, dt = \int_1^{37} u^{\frac{1}{2}}\, du = \frac{2}{3}u^{\frac{3}{2}}\Big|_1^{37} = \frac{2}{3}\left(37^{\frac{3}{2}} - 1\right)$$

$u = 4t^2+1$
$du = 8t\, dt$

(b) If a potential function f exists,

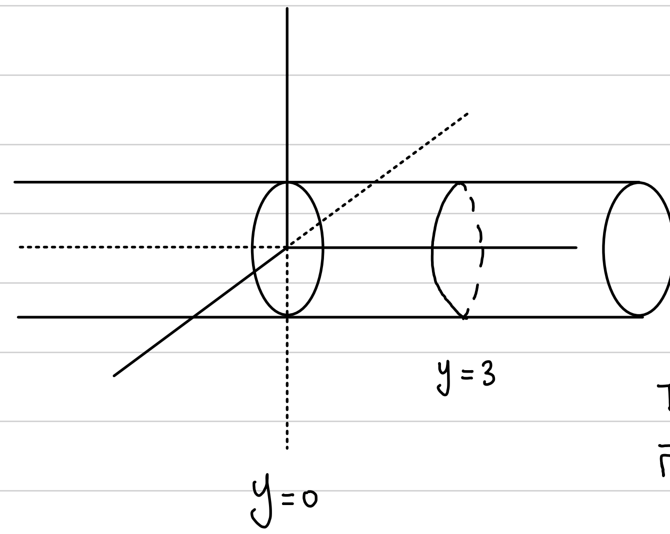
$$\left.\begin{array}{ll} f_x = 1 & f = x + g_1(y,z) \\ f_y = \frac{1}{z} & f = \frac{y}{z} + g_2(x,z) \\ f_z = -\frac{y}{z^2} & f = \frac{y}{z} + g_3(x,y) \end{array}\right\} \Rightarrow f = x + \frac{y}{z} \text{ works}$$

$\vec{r}\left(\frac{1}{2}\right) = \left(\frac{1}{4}, \frac{1}{2}, 2\right)$

$\vec{r}(1) = (1, 0, 1)$

$$\int_C \left(1, \frac{1}{z}, -\frac{y}{z^2}\right) \cdot d\vec{r} = f(1,0,1) - f\left(\frac{1}{4}, \frac{1}{2}, 2\right)$$
$$= 1 - \left(\frac{1}{4} + \frac{1}{4}\right) = \frac{1}{2}$$

2. Set up the integral for the mass of Σ, the part of the cylinder $x^2 + z^2 = 4$ between $y = 0$ and $y = 3$ if the mass density at (x, y, z) is given by $\delta(x, y, z) = x^2$. Proceed until you have an iterated double integral but do not evaluate.



We can use the parametrization

$\vec{r}(\theta, y) = (2\cos\theta,\ y,\ 2\sin\theta)$

w/ $0 \leq \theta \leq 2\pi$ and $0 \leq y \leq 3$

Then

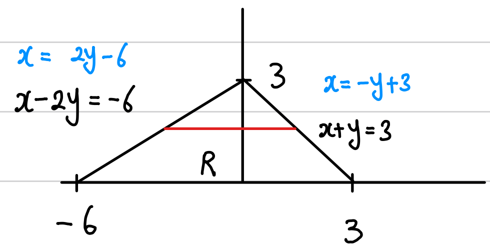
$$\vec{r}_\theta \times \vec{r}_y = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -2\sin\theta & 0 & 2\cos\theta \\ 0 & 1 & 0 \end{vmatrix}$$

$$= (-2\cos\theta,\ 0,\ -2\sin\theta)$$

$$\| \vec{r}_\theta \times \vec{r}_y \| = \sqrt{4\cos^2\theta + 4\sin^2\theta} = 2$$

Then $$\iint_\Sigma \delta \, dS = \int_0^{2\pi} \int_0^3 4\cos^2\theta \cdot 2 \, dy \, d\theta$$

3. Let C be the triangle with corners $(-6, 0)$, $(0, 3)$ and $(3, 0)$ with clockwise orientation. Use Green's Theorem to evaluate $\int_C y^2\ dx + 6xy\ dy$.



$$\int_C y^2 dx + 6xy\, dy = -\iint_R (6y - 2y)\, dA = -4\int_0^3 \int_{2y-6}^{-y+3} y\, dx\, dy$$

$$= -4\int_0^3 y(-y+3-2y+6)\, dy$$

$$= -4\int_0^3 y(-3y+9)\, dy$$

$$= 12\int_0^3 y^2 - 3y\, dy$$

$$= 12\left(\frac{y^3}{3} - \frac{3y^2}{2}\right)\Big|_0^3$$

$$= 12\left(9 - \frac{27}{2}\right)$$

$$= 12\left(\frac{18}{2} - \frac{27}{2}\right) = -\frac{9 \cdot 12}{2} = -9 \cdot 6 = -54$$

4. Let C be the intersection of the parabolic sheet $z = 9 - x^2$ with cylinder $r = \sin\theta$, oriented counterclockwise when viewed from above. Apply Stokes' Theorem to the line integral

$$\int_C (xz\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k}) \cdot d\mathbf{r}$$

Parametrize the resulting surface and proceed until you have an iterated double integral but do not evaluate.

We use the Cartesian coordinate $\vec{r}(x,y) = (x, y, 9-x^2)$ and figure out the bounds later

Then

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -2x \\ 0 & 1 & 0 \end{vmatrix} = 2x\hat{i} + \hat{k}$$

always points north (points outwards)

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & x & y \end{vmatrix} = (1-0)\hat{i} - (0-x)\hat{j} + (1-0)\hat{k}$$

$$= (1, x, 1)$$

Then

$$\int_C \vec{F} \cdot d\vec{r} = \iint_\Sigma \text{curl}\,\vec{F} \cdot \vec{n}\, dS$$

$$= \pm \iint_\Sigma (1,x,1) \cdot (2x,0,1)\, dS$$

$$= \pm \iint_\Sigma 2x+1\, dS$$

$$= \pm \int_0^\pi \int_0^{\sin\theta} (2r\cos\theta + 1)\, r\, dr\, d\theta \quad \text{in polar coordinates}$$

$$= \int_0^\pi \int_0^{\sin\theta} (2r\cos\theta + 1)\, r\, dr\, d\theta \quad \text{as } \vec{r}_x \times \vec{r}_y \text{ points outwards}$$

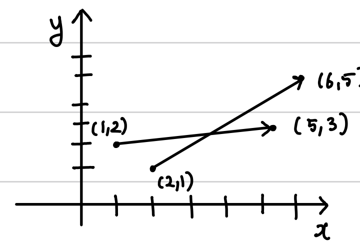
5. (a) Let $f(x, y) = x^2y$. Sketch the vectors for the vector field ∇f at the points $(1, 2)$ and $(2, 1)$.

(b) Evaluate $\iint_{\Sigma} (x\,\mathbf{i} + xz\,\mathbf{j} + 5z\,\mathbf{k}) \cdot \mathbf{n}\; dS$ where Σ is the part of the sphere $x^2 + y^2 + z^2 = 9$ above the xy-plane along with the disk $x^2 + y^2 \le 9$ in the xy-plane. Assume Σ has outwards orientation.

(a) $\nabla f(x,y) = (2xy, x^2)$

$\nabla f(1,2) = (4,1)$

$\nabla f(2,1) = (4,4)$



(b) $\vec{F}(x,y,z) = (x, xz, 5z)$

$\operatorname{div} \vec{F} = 1 + 0 + 5 = 6$

$$\iint_{\Sigma} \vec{F} \cdot \vec{n}\, dS = \iiint_D \operatorname{div} \vec{F} \cdot dV$$

$$= 6 \iiint_D dV = 6 \cdot 18\pi \quad \text{---- by ① or ②}$$

$$= 108\pi$$

① D is a half sphere of radius 3

$$\iiint_D dV = \frac{2\pi}{3} \cdot 3^3 = 18\pi$$

②

$$\iiint_D dV = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^3 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \frac{\rho^3}{3} \sin\phi \Big|_0^3 \, d\phi \, d\theta$$

$$= 9 \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \sin\phi \, d\phi \, d\theta$$

$$= 9 \int_0^{2\pi} -\cos\phi \Big|_0^{\frac{\pi}{2}} \, d\theta$$

$$= 9 \int_0^{2\pi} (0 - (-1)) \, d\theta$$

$$= 18\pi$$