Quiz 8

Math 241: Spring 2023 **Answer Key**

Problem 1. (5 points) Find the curl of $\mathbf{F} = \left(y^2 + \cos z\right)\mathbf{i} + 2xy\mathbf{j} + e^x\mathbf{k}$. Is this vector field conservative? Explain your answer. (You do **not** need to compute a potential function.)

We compute that

$$\operatorname{curl}\mathbf{F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}$$

$$= \left(\frac{\partial}{\partial y}\left(e^x\right) - \frac{\partial}{\partial z}\left(2xy\right)\right)\mathbf{i} + \left(\frac{\partial}{\partial z}\left(y^2 + \cos z\right) - \frac{\partial}{\partial x}\left(e^x\right)\right)\mathbf{j} + \left(\frac{\partial}{\partial x}\left(2xy\right) - \frac{\partial}{\partial y}\left(y^2 + \cos z\right)\right)\mathbf{k}$$

$$= (0-0)\,\mathbf{i} + (-\sin z - e^x)\,\mathbf{j} + (2y - 2y)\,\mathbf{k} = \boxed{(-\sin z - e^x)\,\mathbf{j}}.$$

The vector field $\mathbf{F}$ is not conservative, because $\operatorname{curl}\mathbf{F} \neq \mathbf{0}$.

Problem 2. (5 points) Compute the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F} = 3x\mathbf{i} - y\mathbf{j}$ and C is the portion of the parabola $y = x^2$ with $-1 \le x \le 2$.

We parametrize C by $\mathbf{r}(t) = (t, t^2)$, with $-1 \le t \le 2$. With that, $\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}$. We thus compute

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{-1}^{2} \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = \int_{-1}^{2} \left\langle 3t, -t^2\right\rangle \cdot \langle 1, 2t\rangle\ dt$$

$$= \int_{-1}^{2} (3t)\,(1) + \left(-t^2\right)(2t)\ dt = \int_{-1}^{2} 3t - 2t^3\,dt = \frac{3}{2}t^2 - \frac{1}{2}t^4\Big|_{-1}^{2}$$

$$= \left(\frac{3}{2}(2)^2 - \frac{1}{2}(2)^4\right) - \left(\frac{3}{2}(-1)^2 - \frac{1}{2}(-1)^4\right) = \boxed{-3}.$$