Thursday Practice 1 Solution
MATH 241 (Spring 2023)
02/02/2023

Problem 1. We denote by

$$P = (1, 0, 3), \qquad Q = (0, -2, 1), \qquad R = (3, 2, 1)$$

Then the three sides of the triangle are

$$\begin{aligned}
\overrightarrow{PQ} &= Q - P &&= (0, -2, 1) - (1, 0, 3) &&= (-1, -2, -2) \\
\overrightarrow{PR} &= R - Q &&= (3, 2, 1) - (1, 0, 3) &&= (2, 2, -2) \\
\overrightarrow{QR} &= R - Q &&= (3, 2, 1) - (0, -2, 1) &&= (3, 4, 0)
\end{aligned}$$

Then

$$\begin{aligned}
||\overrightarrow{PQ}|| &= \sqrt{(-1)^2 + (-2)^2 + (-2)^2} &&= 3 \\
||\overrightarrow{QR}|| &= \sqrt{2^2 + 2^2 + (-2)^2} &&= 2\sqrt{3} \\
||\overrightarrow{QR}|| &= \sqrt{3^2 + 4^2 + 0^2} &&= 5
\end{aligned}$$

Hence the perimeter of the triangle is $8 + 2\sqrt{3}$.

Problem 2.

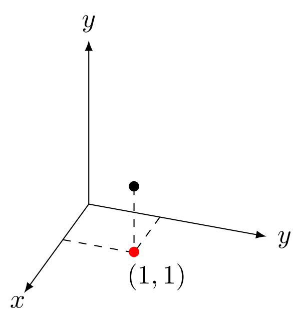
$$(x + 2)^2 + y^2 + (z - 7)^2 \leq 36$$

Problem 3.

(a) Note that the figure is a unit cube, so each side has length 1.

$$\overrightarrow{OQ} = (1, 1, 1) \qquad \text{and} \qquad \overrightarrow{OR} = \left(\frac{1}{2}, 1, \frac{1}{2}\right)$$

To see this, we can first project the point to the xy-plane. For example, the projection of $\overrightarrow{OQ}$ to the xy-plane is $(1, 1)$ given by a red point below.



Tracing back to $\overrightarrow{OQ}$ along the vertical line, one sees that the vector $\overrightarrow{OQ}$ is given by $(1, 1, 1)$.

(b) Recall the equation $\overrightarrow{OQ} \cdot \overrightarrow{OR} = ||\overrightarrow{OQ}||||\overrightarrow{OR}|| \cos\theta$ where θ is the angle between $\overrightarrow{OQ}$ and $\overrightarrow{OR}$. Then

$$\cos\theta = \frac{\overrightarrow{OQ} \cdot \overrightarrow{OR}}{||\overrightarrow{OQ}||||\overrightarrow{OR}||} = \frac{2}{\sqrt{3}\sqrt{\frac{3}{2}}} = \frac{2}{3/\sqrt{2}} = \frac{2\sqrt{2}}{3}$$

Problem 4.

(a)

$$\boldsymbol{v} * (\boldsymbol{w}_1 + \boldsymbol{w}_2) \overset{(i)}{=} (\boldsymbol{w}_1 + \boldsymbol{w}_2) * \boldsymbol{v} \overset{(iv)}{=} (\boldsymbol{w}_1 * \boldsymbol{v}) + (\boldsymbol{w}_2 * \boldsymbol{v}) \overset{(i)}{=} (\boldsymbol{v} * \boldsymbol{w}_1) + (\boldsymbol{v} * \boldsymbol{w}_2)$$

(b)

$$\begin{aligned} ||\boldsymbol{v}+\boldsymbol{w}||^2 &= (\boldsymbol{v}+\boldsymbol{w}) * (\boldsymbol{v}+\boldsymbol{w}) \\ &= (\boldsymbol{v}+\boldsymbol{w}) * \boldsymbol{v} + (\boldsymbol{v}+\boldsymbol{w}) * \boldsymbol{w} \\ &= \boldsymbol{v} * \boldsymbol{v} + \boldsymbol{v} * \boldsymbol{w} + \boldsymbol{w} * \boldsymbol{v} + \boldsymbol{w} * \boldsymbol{w} \\ &= ||\boldsymbol{v}||^2 + ||\boldsymbol{w}||^2 + 2(\boldsymbol{v} * \boldsymbol{w}) \end{aligned}$$

(c) By manipulating (b), we see that

$$v * w = \frac{1}{2}(||\boldsymbol{v}+\boldsymbol{w}||^2 - ||\boldsymbol{v}||^2 + ||\boldsymbol{w}||^2)$$

We have

$$\begin{aligned} ||\boldsymbol{v}+\boldsymbol{w}||^2 &= (a_1+a_2)^2 + (b_1+b_2)^2 + (c_1+c_2)^2 \\ &= a_1^2 + 2a_1a_2 + a_2^2 + b_1^2 + 2b_1b_2 + b_2^2 + c_1^2 + 2c_1c_2 + c_2^2 \\ &= (a_1^2 + b_1^2 + c_1^2) + (a_2^2 + b_2^2 + c_2^2) + 2a_1a_2 + 2b_1b_2 + 2c_1c_2 \end{aligned}$$

As $||\boldsymbol{v}||^2 = a_1^2 + b_1^2 + c_1^2$ and $||\boldsymbol{w}||^2 = a_2^2 + b_2^2 + c_2^2$, it is now not too difficult to see

$$||\boldsymbol{v}+\boldsymbol{w}||^2 - ||\boldsymbol{v}||^2 - ||\boldsymbol{w}||^2 = 2a_1a_2 + 2b_1b_2 + 2c_1c_2$$

In conclusion, we have

$$\boldsymbol{v} * \boldsymbol{w} = \frac{1}{2}(2a_1a_2 + 2b_1b_2 + 2c_1c_2) = a_1a_2 + b_1b_2 + c_1c_2 = \boldsymbol{v} \cdot \boldsymbol{w}$$