Thursday Practice 3 Solutions
MATH 241 (Spring 2023)
02/16/2023

Problem 1. All functions in the derivative are continuous, so we only check that second condition that $\boldsymbol{r}'(t) \neq 0$.

(a) The derivative is
$$\boldsymbol{r}'(t) = (2t, 18t^5, 0)$$
Then $\boldsymbol{r}'(0) = 0$, so $\boldsymbol{r}$ is not smooth.

(b) The derivative is
$$\boldsymbol{r}'(t) = (\sin t, 0, \cos t)$$
There is no single t such that $\sin t = \cos t = 0$. This is because $\sin t = 0$ if and only if $t = \frac{k\pi}{2}$ for k even integer, and $\cos t = 0$ if and only if $t = \frac{k\pi}{2}$ for k odd integer. There is no integer that is both odd and even. This shows that $\boldsymbol{r}'(t) \neq \mathbf{0}$ for any t. Hence $\boldsymbol{r}$ is smooth.

(c) The derivative is
$$\boldsymbol{r}'(t) = (10t, 3t^2, \frac{1}{t})$$
Notice that $\frac{1}{t}$ is never zero and not defined at $t = 0$. Therefore, $\boldsymbol{r}'(t) \neq \mathbf{0}$ for all $t > 0$ and $\boldsymbol{r}$ is smooth.

Problem 2.

(a) The line parallel to the line is $(-8, 10, 7) - (2, 4, -3) = (-10, 6, 10)$. Then the parametrization is given by
$$\boldsymbol{r}(t) = (-10t + 2)\hat{\boldsymbol{\imath}} + (6t + 4)\hat{\boldsymbol{\jmath}} + (10t - 3)\hat{\boldsymbol{k}}$$
for $0 \leq t \leq 1$.

(b) We first figure out the radius of the semicircle. We have
$$\left|\left|\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)\right|\right| = \left|\left|\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)\right|\right| = 1$$
Therefore, the semi-circle has a raidus of 1. We now solve
$$\cos t = \frac{\sqrt{2}}{2} \ \ \& \ \ \sin t = \frac{\sqrt{2}}{2} \quad \text{and} \quad \cos t = -\frac{\sqrt{2}}{2} \ \ \& \ \ \sin t = -\frac{\sqrt{2}}{2}$$
Then we have $t = \frac{\pi}{4}$ and $t = \frac{5\pi}{4}$. The parametrization is
$$\boldsymbol{r}(t) = \cos t\hat{\boldsymbol{\imath}} + \sin t\hat{\boldsymbol{\jmath}}$$
for $\frac{\pi}{4} \leq t \leq \frac{5\pi}{4}$ because it passes through the second quadrant.

Problem 3. The derivative is
$$\boldsymbol{r}'(t) = (2\cos t, -2\sin t, 0)$$
Hence the arc length
$$\int_0^\pi ||\boldsymbol{r}'(t)||\, dt = \int_0^\pi \sqrt{4\cos^2 t + 4\sin^2 t}\, dt = \int_0^\pi 2\, dt = 2\pi$$