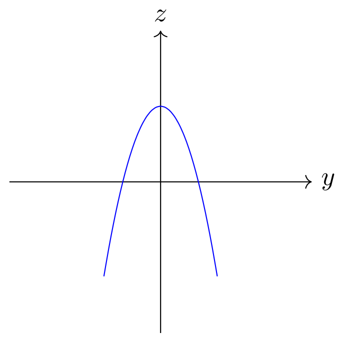
Thursday Practice 5 Solutions
MATH 241 (Spring 2023)
03/09/2023

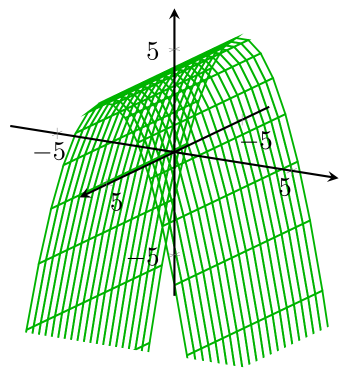
Problem 1.

(a) In the yz-plane, we have the parabola

$$z = 4 - y^2$$



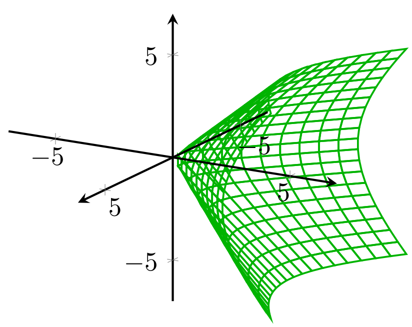
Since x can be any value, we have the the following graph by streching along the x-axis.



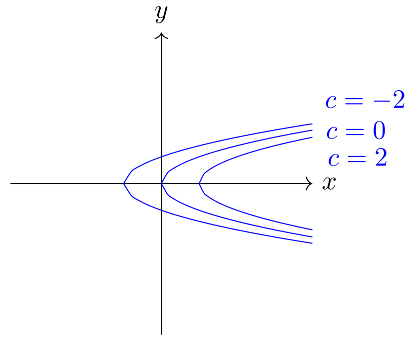
(b) If we slice the cone opening around the positive y-axis by xz-plane, we get a circle as a section. Therefore, the right-hand side should be $x^2 + z^2$. Since it starts at the origin, the equation is

$$y = \sqrt{x^2 + z^2}$$

for $y \geq 0$.



(c) We have $c = x - y^2 \Rightarrow x = y^2 + c$. Then



Problem 2.

(a)

We first find the unit vector in the direction of $1\hat{\imath} - 1\hat{\jmath}$. The norm $||\hat{\imath} - \hat{\jmath}|| = \sqrt{2}$, so we use $\boldsymbol{u} = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$. Next, we compute

$$\nabla f(x,y) = (f_x(x,y), f_y(x,y)) = \left(\frac{2x}{y^3}, -\frac{3x^2}{y^4}\right)$$

Hence $\nabla f(2,-1) = (-4,-12)$.

$$D_{\boldsymbol{u}} f(2,-1) = \nabla f(2,-1) \cdot \boldsymbol{u} = -\frac{4}{\sqrt{2}} + \frac{12}{\sqrt{2}} = \frac{8}{\sqrt{2}} = 4\sqrt{2}$$

(b) We need to find $\frac{dR}{dt}$ given that

$$\frac{dV}{dt} = -2, \qquad \frac{dP}{dt} = -3, \qquad V = 10, \qquad R = 15$$

Using chain rule, we get

$$\frac{dP}{dt} = \frac{\partial P}{\partial V}\frac{dV}{dt} + \frac{\partial P}{\partial R}\frac{dR}{dt} = \frac{2V}{R}\frac{dV}{dt} - \frac{V^2}{R^2}\frac{dR}{dt}$$

Then

$$-3 = \frac{20}{15} \cdot (-2) - \frac{10^2}{15^2} \cdot \frac{dR}{dt} = -\frac{8}{3} - \frac{4}{9}\frac{dR}{dt}$$

Then one has

$$9 = 8 + \frac{4}{3}\frac{dR}{dt} \Rightarrow 1 = \frac{4}{3}\frac{dR}{dt} \Rightarrow \frac{dR}{dt} = \frac{3}{4}$$

Problem 3.

(a) We use the function $f(x,y) = \sqrt{x^2 - y}$. Then

$$f_x(x,y) = \frac{2x}{2\sqrt{x^2-y}} = \frac{x}{\sqrt{x^2-y}} \qquad \text{and} \qquad f_y(x,y) = -\frac{1}{2\sqrt{x^2-y}}$$

We find the equation of the tangent plane at $(3,5)$ which is

$$\begin{aligned} z &= f(3,5) + f_x(3,5)(x-3) + f_y(3,5)(y-5) \\ &= 2 + \frac{3}{2}(x-3) - \frac{1}{4}(y-5) \end{aligned}$$

Hence $\sqrt{2.99^2 - 4.97} \approx 2 + \frac{3}{2} \cdot (-0.01) - \frac{1}{4}(-0.03) = 2 - \frac{3}{200} + \frac{3}{400} = \frac{797}{400}$

(b) Define $F(x,y,z) = y^3 - z^2 - x$. We have

$$F_x(x,y,z) = -1, \qquad F_y(x,y,z) = 3y^2, \qquad F_z(x,y,z) = -2z$$

Then the plane tangent to the surace at $(-1, 2, 3)$ is given by

$$\begin{aligned} 0 &= F_x(-1,2,3)(x+1) + F_y(-1,2,3)(y-2) + F_z(-1,2,3)(x-3) \\ &= -(x+1) + 12(y-2) - 6(z-3) \end{aligned}$$

Problem 4.

(a)

$$f_x(x,y) = 2xy - 2x \qquad \text{and} \qquad f_y(x,y) = x^2 - 4y$$

Then $f_x(x,y) = 2xy - 2x = 0 \Rightarrow xy - x = 0 \Rightarrow x(y-1) = 0$. Then $x = 0$ or $y = 1$.

(a) $x = 0$. Then $f_y(x,y) = x^2 - 4y = 0 \Rightarrow -4y = 0 \Rightarrow 0$.

(b) $y = 1$. Then $f_y(x,y) = x^2 - 4y = 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$.

Hence the only critical points are $(0,0)$, $(2,1)$, and $(-2,1)$.

(b)

$$f_{xx}(x,y) = 2y - 2, \qquad f_{xy}(x,y) = 2x, \qquad f_{yx}(x,y) = 2x, \qquad f_{yy}(x,y) = -4$$

Then $D(x,y) = -4(2y-2) - 4x^2$.

	$D(x,y)$	$f_{xx}(x,y)$	
$(0,0)$	$8 > 0$	$-2 < 0$	relative maximum
$(2,1)$	$-16 < 0$	0	saddle point
$(-2,1)$	$-16 < 0$	0	saddle point

Problem 5. Let $g(x,y) = x^2 + y^2$. Then we solve

$$\begin{cases} 4x = \lambda 2x \\ 2y^2 = \lambda 2y \end{cases} \Rightarrow \begin{cases} 2x = \lambda x \\ y^2 = \lambda y \end{cases}$$

(a) Let $x = 0$. Then by the constraint $x^2 + y^2 = 4$, $y = \pm 2$.

(b) Let $x \neq 0$, then Then $2x = \lambda x \Rightarrow \lambda = 2$. Then $y^2 = 2y \Rightarrow y^2 - 2y = 0 \Rightarrow y(y-2) = 0$. So $y = 0$ or $y = 2$. If $y = 2 \Rightarrow x = 0$ by the constraint, so $y = 0$. We can conclude that $x = \pm 2$ and $y = 0$.

$$f(\pm 2, 0) = 8, \qquad f(0,2) = \frac{16}{3}, \qquad f(0,-2) = -\frac{16}{3}$$

Therefore, maximum is 8 and minimum is $-\frac{16}{3}$.