Thursday Practice 7	**Name:**	____________________
MATH 241 (Spring 2023)	**Section:**	0112 (8AM-9AM) / 0122 (9:30AM-10:20AM)
04/06/2023	**TA:**	Shin Eui Song

Problem 1. Find the volume of the solid region bounded by the plane $x + 3y + 6z = 1$, $x = 0$, $y = 0$, and $z = 0$.

Problem 2. Find the volume of the solid region bounded above by the sphere $x^2 + y^2 + z^2 = 2$ and below by the circular paraboloid $z = x^2 + y^2$ using the cylindrical coordinate.

Problem 3. Find the volume of the solid region between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$ and below the upper nappe of the cone $3z^2 = x^2 + y^2$ using the spherical coordiante.

Problem 4. (Challenge Problem) Find the volume of the solid region bounded above by the sphere $x^2 + y^2 + z^2 = 64$ and below by the plane $z = -4\sqrt{3}$ using the spherical coordinate. The following are just suggestions to solve the problem.

(a) Find volume V_1 which is bounded above by the plane and bounded below by the sphere. Then the volume V we want to find is

$$V = (\text{volume of the sphere}) - V_1 = 4\pi 8^3 - V_1$$

(b) Compute ϕ when the plane and the sphere intersect by chaing both equations into spherical coordinate.

(c) Express the range of ρ in terms of ϕ.

Thursday Practice 7 Solutions
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Problem 1. Let R be the region formed by projecting the solid onto the xy-plane. This is given by the line $x + 3y = 1$, or equivalently, $y = \frac{1}{3}(1 - x)$. The *top function* is given by $z = \frac{1}{6}(1 - x - 3y)$ and the *bottom function* is given by $z = 0$, so the integral is

$$\begin{aligned}
\int_0^1 \int_0^{\frac{1}{3}(1-x)} \int_0^{\frac{1}{6}(1-x-3y)} dz\ dy\ dx &= \int_0^1 \int_0^{\frac{1}{3}(1-x)} \frac{1}{6}(1-x) - \frac{1}{2}y\ dy\ dx \\
&= \int_0^1 \frac{1}{6}(1-x)y - \frac{1}{4}y^2 \Big|_0^{\frac{1}{3}(1-x)} dx \\
&= \int_0^1 \frac{1}{18}(1-x)^2 - \frac{1}{36}(1-x)^2\ dx \\
&= \int_0^1 \frac{1}{36}(1-x)^2\ dx \\
&= \int_1^0 -\frac{1}{36}u^2\ du \\
&= -\frac{1}{108}u^3 \Big|_1^0 = \boxed{\frac{1}{108}}
\end{aligned}$$

Problem 2. If we substitute $z = x^2 + y^2$ into the sphere equation $x^2 + y^2 + z^2 = 2$, we get $z + z^2 = 2$. This factors as $(z + 2)(z - 1) = 0$, hence $z = -2$ or $z = 1$. Based on our sketch, $z > 0$, so $z = 1$. This says that the intersection between the sphere and the paraboloid is $x^2 + y^2 = 1$ which is a circle of radius 1. Then the integral is

$$\int_0^{2\pi} \int_0^1 \int_?^? r\ dz\ dr\ d\theta$$

To figure out the range for z, we convert the equations into cylindrical coordinates. Namely, the paraboloid is $z = x^2 = y^2 = r^2$. For sphere, first solve for z. Then $z^2 = 2 - r^2 \Rightarrow z = \pm\sqrt{2 - r^2}$. Again, $z \geq 0$ in the sketch, so $z = \sqrt{2 - r^2}$. Since the *top function* is the sphere and the *bottom function* is the paraboloid, the integral is

$$\int_0^{2\pi} \int_0^1 \int_{r^2}^{\sqrt{2-r^2}} r\ dz\ dr\ d\theta$$