Thursday Practice 8	**Name:** ____________________
MATH 241 (Spring 2023)	**Section:** 0112 (8AM-9AM) / 0122 (9:30AM-10:20AM)
04/20/2023	**TA:** Shin Eui Song

Checklist

- curl and divergence of a vector field, and conservativity
- line integral of scalar valued function
- line integral of vector field

Problem 1. Evaluate $\int_C xy^4 \, ds$ where C is the right half of the circle $x^2 + y^2 = 16$ with counter-clockwise orientation.

Problem 2. Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x, y) = (x + y, 1 - x)$ and C is the portion of $\frac{x^2}{4} + \frac{y^2}{9} = 1$ that is in the 4th quadrant with the counter-clockwise orientation. (Hint: use the fact that $2\sin(t)\cos(t) = 2\sin(2t)$.)

Problem 3. Find the curl and the divergence of $\vec{F}(x, y, z) = (e^x \cos y, e^x \sin y, z)$.

Thursday Practice 8 Solutions
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Problem 1. The parametrization of the right half of the circle $x^2 + y^2 = 16$ is given by

$$\boldsymbol{r}(t) = (4\cos t, 4\sin t)$$

for $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$. Then $\boldsymbol{r}'(t) = (-4\sin t, 4\cos t)$ and $||\boldsymbol{r}'(t)|| = \sqrt{16\sin^2 t + 16\cos^2 t} = 4$. Therefore, the line integral

$$\int_C xy^4 \, ds = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (4\cos t)(4\sin t)^4 \cdot 4dt = 4^6 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos t \sin^4 t \, dt = (*)$$

Set $u = \sin t$, then

$$(*) = 4^6 \int_{-1}^{1} u^4 \, du = 4^6 \cdot \left. \frac{u^5}{5} \right|_{-1}^{1} = \boxed{\frac{2 \cdot 4^6}{5}}$$

Problem 2. The parametrization is given by $\vec{r}(t) = (2\cos t, 3\sin t)$ with $-\frac{\pi}{2} \le t \le 0$. Therefore

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= \int_{-\frac{\pi}{2}}^{0} (2\cos t + 3\sin t, 1 - 2\cos t) \cdot (-2\sin t, 3\cos t) \, dt \\
&= \int_{-\frac{\pi}{2}}^{0} -4\cos t \sin t - 6\sin^2 t + 3\cos t - 6\cos^2 t \, dt \\
&= \int_{-\frac{\pi}{2}}^{0} -2\sin(2t) + 3\cos t - 6 \, dt \\
&= \left. \cos(2t) + 3\sin(t) - 6t \right|_{-\frac{\pi}{2}}^{0} \\
&= (1 + 0 - 0) - (-1 - 3 + 3\pi) = \boxed{5 - 3\pi}
\end{aligned}$$

Problem 3.

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ e^x \cos y & e^x \sin y & z \end{vmatrix} = (0 - 0)\hat{\boldsymbol{\imath}} - (0 - 0)\hat{\boldsymbol{\jmath}} + (e^x \sin y - (-e^x \sin y))\hat{\boldsymbol{k}} = (2e^x \sin y)\hat{\boldsymbol{k}}$$

$$\text{div } \vec{F} = \frac{\partial}{\partial x}(e^x \cos y) + \frac{\partial}{\partial y}(e^x \sin y) + \frac{\partial}{\partial z}(z) = e^x \cos y + e^x \cos y + 1 = 2e^x \cos y + 1$$