Thursday Practice 9		**Name:**	____________________
MATH 241 (Spring 2023)		**Section:**	0112 (8AM-9AM) / 0122 (9:30AM-10:20AM)
04/27/2023		**TA:**	Shin Eui Song

Problem 1. Compute the following line integral

$$\frac{1}{2}\int_C -y\ dx + x\ dy$$

where C is an ellipse given by $\frac{x^2}{4} + y^2 = 1$. (**Hint:** It is easier to directly compute the line integral.)

Problem 2. Compute the following line integral

$$\int_C (5x^2 + 4)\ dx + x\ dy$$

where C is a circle of raidus 3.

Problem 3. Evaluate

$$\iint_{\Sigma} x \; dS$$

where Σ is the part of the plane $2x + 3y + z = 6$ in the first octant.

Problem 4. Show that the line integral

$$\int_C (\cos x + 2yz) \; dx + (\sin y + 2xz) \; dy + (z + 2xy) \; dz$$

is independent of path, and evaluate the integral when C is parametrized by $\boldsymbol{r}(t) = (t, t, \pi^{-2}t)$ for $0 \leq t \leq \pi$.

Thursday Practice 9 Solutions
MATH 241 (Spring 2023)
04/27/2023

Problem 1. Let $\boldsymbol{r}(t) = (2\cos t, \sin t)$ be a parametrization of the ellipse for $0 \le t \le 2\pi$. Then $\boldsymbol{r}(t) = (-2\sin t, \cos t)$. Plugging into the line integral, we obtain

$$\frac{1}{2}\int_0^{2\pi} (-\sin t)(-2\sin t) + (2\cos t)(\cos t) = \frac{1}{2}\int_0^{2\pi} 2\ dt = \boxed{2\pi}$$

Problem 2. We have $P = 5x^2 + 4$ and $Q = x$, so

$$\int_C (5x^2+4)\ dx + x\ dy = \iint_R 1\ dA$$

Then one immediately knows that the area of the circle of radius 3 is 9π. Or, we change to polar coordinate to obtain

$$\int_0^{2\pi}\int_0^3 r\ dr\ d\theta = \int_0^{2\pi} \frac{r}{2}\Big|_0^3\ d\theta = \int_0^{2\pi} \frac{9}{2}\ d\theta = \boxed{9\pi}$$

Problem 3. We use the parametrization $\boldsymbol{r}(x,y) = (x, y, 6-2x-3y)$. Then the region R in the xy-plane is given by

$$\left\{(x,y)\ \middle|\ 0 \le x \le 3 \text{ and } 0 \le y \le 2 - \frac{2}{3}x\right\}$$

We have

$$\boldsymbol{r}_x = (1,0,-2) \text{ and } \boldsymbol{r}_y = (0,1,-3)$$

so

$$||\boldsymbol{r}_x \times \boldsymbol{r}_y|| = ||\begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ 1 & 0 & -2 \\ 0 & 1 & -3 \end{vmatrix}|| = ||(2,3,1)|| = \sqrt{14}$$

Hence the integral becomes

$$\int_0^3\int_0^{2-\frac{2}{3}x} x\sqrt{14}\ dy\ dx = \sqrt{14}\int_0^3 xy\Big|_0^{2-\frac{2}{3}x}\ dx = \sqrt{14}\int_0^3 x(2-\frac{2}{3}x)\ dx = \sqrt{14}\int_0^3 2x - \frac{2}{3}x^2\ dx$$

As

$$\int_0^3 2x - \frac{2}{3}x^2\ dx = x^2 - \frac{2}{9}x^3\Big|_0^3 = 9 - 6 = 3$$

We have that

$$\iint_\Sigma x\ dS = \sqrt{14}\int_0^3 2x - \frac{2}{3}x^2\ dx = \boxed{3\sqrt{14}}$$

Problem 4. If the vector field is conservative, there exists an f such that

$$f_x = \cos x + 2yz \text{ and } f_y = \sin y + 2xz \text{ and } f_z = z + 2xy$$

Then

$$f = \int f_x\ dx = \sin x + 2xyz + g(y,z)$$

Then

$$f_y = 2xz + \frac{\partial g}{\partial y} = \sin y + 2xz$$

Hence $\frac{\partial g}{\partial y} = \sin y \Rightarrow g = -\cos y + h(z)$. Therefore, $f = \sin x + 2xyz - \cos y + h(z)$. Again taking the derivative in terms of z, we get $f_z = 2xy + h'(z) = z + 2xy \Rightarrow h'(z) = z \Rightarrow h(z) = \frac{z^2}{2} + C$. Since we can choose any constant, let $C = 0$. Thus, we have obtained a potential function

$$f = \sin x + 2xyz - \cos y + \frac{z^2}{2}$$

Note that the starting point of $\boldsymbol{r}$ is $(0,0,0)$ and the ending point is $(\pi, \pi, \frac{1}{\pi})$. Then we have that the integral is equal to

$$f(\pi, \pi, \frac{1}{\pi}) - f(0,0,0) = 2\pi + 1 + \frac{1}{2\pi^2} - (-1) = 2\pi^2 + \frac{1}{2\pi^2}$$