Practice Problems
Exam 2 Solution

MATH 240

Problem 1 $A^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Problem 2 (a) $\underbrace{\begin{bmatrix} 3 & 1 \\ 4 & -2 \end{bmatrix}}_{=A}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11 \\ 14 \end{bmatrix}$ Then $\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}\begin{bmatrix} 11 \\ 14 \end{bmatrix} \overset{\text{Problem 1}}{=} \frac{1}{-6-4}\begin{bmatrix} -2 & -1 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} 11 \\ 14 \end{bmatrix}$

$$= \frac{1}{-10}\begin{bmatrix} -22-14 \\ -44+42 \end{bmatrix} = -\frac{1}{10}\begin{bmatrix} -36 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{18}{5} \\ \frac{1}{5} \end{bmatrix}$$

(b) Let $A = \begin{bmatrix} 2 & 1 & 4 \\ 0 & -1 & 1 \\ 0 & 1 & -2 \end{bmatrix}$. Then we can find A^{-1} by

$$[A \mid I_3] = \left[\begin{array}{ccc|ccc} 2 & 1 & 4 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 \end{array}\right] \underset{\text{step skipped}}{\longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{3}{2} & 1 \\ 0 & 1 & 0 & 0 & -2 & -1 \\ 0 & 0 & 1 & 0 & -1 & -1 \end{array}\right] \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{3}{2} & 1 \\ 0 & -2 & -1 \\ 0 & -1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 1 \end{bmatrix}$$

$$\underbrace{\begin{matrix} \frac{1}{2} & \frac{3}{2} & 1 \\ 0 & -2 & -1 \\ 0 & -1 & -1 \end{matrix}}_{=A^{-1}}$$

Problem 3 (a) False

Consider $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\}$ Any two are not scalar multiples of each other, but not a basis of $\mathbb{R}^3$, as $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \notin \operatorname{Span} S$.

(b) True

$$A = \underbrace{\left[\begin{array}{c|c} \overbrace{I_4}^{} & \overbrace{0}^{4} \\ \hline \multicolumn{2}{c}{0} \end{array}\right]}_{8} \} 4$$

is a 5×8 matrix w/ rank 4.

(c) False

$$A = \underbrace{\left[\begin{array}{c} I_5 \\ \hline 0 \end{array}\right]}_{5} \begin{matrix} \} 5 \\ \} 5 \end{matrix}$$

is a 10×5 matrix w/ rank 5

By rank theorem, nullity = # of cols − rank

$= 5 - 5 = 0 < 2$

(d) False

Consider $3 + x = a(1+x) + b(2+x)$

$= (a+2b) + (a+b)x$

$\Rightarrow \begin{cases} a + 2b = 3 \\ a + b = 1 \end{cases}$ Then $b = 2,\ a = -1$

Hence the set is linearly dependent

(e) True

$\ker(A) = \{\text{Solution set of } Ax = 0\}$

$\underset{\text{By row equivalence}}{=} \{\text{Solution set of } Bx = 0\} = \ker(B)$

Problem 4

$\text{range}(T) = \{2a \mid a \in \mathbb{R}\} = \{a \mid a \in \mathbb{R}\} = \mathbb{P}_0$ $(=\mathbb{R})$ basis : $\{1\}$ in fact $\{a\}$ with real number $a \neq 0$ works as range(T) has dimension 1

$\ker(T) = \{c+bx+ax^2 \in \mathbb{P}_2 \mid T(c+bx+ax^2) = 0, \; a,b,c \in \mathbb{R}\}$

$= \{c+bx+ax^2 \in \mathbb{P}_2 \mid 2a = 0, \; a,b,c \in \mathbb{R}\}$

$= \{c+bx \in \mathbb{P}_2 \mid c,b \in \mathbb{R}\} = \text{Span}\{1, x\}$ basis : $\{1, x\}$

$\dim \text{Range}(T) + \dim \ker(T) = 2+1 = 3 = \dim \mathbb{P}_2$ verifies rank-nullity theorem

domain

Problem 5

① range $(T) \subseteq \mathbb{R}$, so $\dim \text{range}(T) \leq 1$. Since range (T) contains nonzero vector,

$\dim \text{range}(T) = 1$, so $\{1\}$ is a basis.

② For $\ker(T)$, consider the standard matrix $A = [T(e_1) \; T(e_2) \; T(e_3)] = [1 \; 1 \; 1]$

and describe Null A. See $[1 \; 1 \; 1 \mid 0]$ augmented matrix associated to $Ax = 0$

The general solution is $\begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis.

$\dim \text{range}(T) + \dim \ker(T) = 1+2 = 3 = \dim \mathbb{R}^3$ verifies rank-nullity theorem

Problem 6

By inspection, $6x^2+4 = 2 \cdot (3x^2) + 0 \cdot (x-1) + 1 \cdot 4 \Rightarrow [6x^2+4]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

Problem 7

- $\dim \text{Col}(A) \leq 2$ Therefore $\left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \end{bmatrix} \right\}$ is a basis of Col(A) and $\dim \text{Col}(A) = 2$ because they are not scalar multiplication of each other.
- $\dim \text{Row}(A) = \dim \text{Col}(A) = 2. \Rightarrow \left\{ \begin{bmatrix} 1 \\ 4 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \\ 2 \\ -3 \end{bmatrix} \right\}$ is a basis bc they are not scalar multiple of each other

Problem 8 From the standard basis $\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$, we add to $\left\{ \begin{bmatrix}-1\\0\\2\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix} \right\}$ to make it into a basis.

Try $\begin{bmatrix}1\\0\\0\end{bmatrix}$

$$\begin{bmatrix}-1 & 0 & 1\\ 0 & 1 & 0\\ 2 & 1 & 0\end{bmatrix} \to \begin{bmatrix}-1 & 0 & 1\\ 0 & 1 & 0\\ 0 & 1 & 2\end{bmatrix} \to \begin{bmatrix}-1 & 0 & 1\\ 0 & 1 & 0\\ 0 & 0 & 2\end{bmatrix} \Rightarrow \left\{ \begin{bmatrix}-1\\0\\2\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix} \right\} \text{ is LI w/ 3 elements}$$

$\Rightarrow$ is a basis

Problem 9 S is not closed under addition. $\begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\-1\\0\\0\end{bmatrix} \in S$, but $\begin{bmatrix}1\\1\\0\\0\end{bmatrix} + \begin{bmatrix}1\\-1\\0\\0\end{bmatrix} = \begin{bmatrix}2\\0\\0\\0\end{bmatrix} \notin S$ as $2^2 \neq 0^2$.

Problem 10 We want to show that S is a subspace

① $5\cdot 0 - 4\cdot 0 + 7\cdot 0 = 0 \Rightarrow (0,0,0) \in S$

② Let $(a_1,b_1,c_1), (a_2,b_2,c_2) \in S$,

$$5(a_1+a_2) - 4(b_1+b_2) + 7(c_1+c_2) = (5a_1 - 4b_1 + 7c_1) + (5a_2 - 4b_2 + 7c_2)$$
$$= 0 + 0 = 0$$

$\Rightarrow (a_1+a_2, b_1+b_2, c_1+c_2) \in S$

③ Let $d \in \mathbb{R}$ and $(a,b,c) \in S$. Then

$$5(da) - 4(db) + 7(dc) = d(5a - 4b + 7c) = d\cdot 0 = 0$$

$\Rightarrow (da, db, dc) \in S$

Problem 11

(a) $\left[\begin{array}{cc|cc} -2 & 5 & 1 & -3\\ 4 & -1 & 7 & -3\end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & 2 & -1\\ 0 & 1 & 1 & -1\end{array}\right]$

Step skipped

$\underbrace{\begin{matrix}2 & -1\\ 1 & -1\end{matrix}}_{= \underset{B_2 \leftarrow B_1}{P}}$

(c) $(-7,5) = 1\cdot(-2,4) + (-1)\cdot(5,-1)$

by trial-and-error. Then

$$[(-7,5)]_{B_2} = \begin{bmatrix}1\\-1\end{bmatrix}$$

(b) $\underset{B_1 \leftarrow B_2}{P} = \left(\underset{B_2 \leftarrow B_1}{P}\right)^{-1} = \dfrac{1}{-1}\begin{bmatrix}-1 & 1\\ -1 & 2\end{bmatrix} = \begin{bmatrix}1 & -1\\ 1 & -2\end{bmatrix}$

(d) $[(-7,5)]_{B_1} = \underset{B_1 \leftarrow B_2}{P}[(-7,5)]_{B_2}$

$$= \begin{bmatrix}1 & -1\\ 1 & -2\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}2\\3\end{bmatrix}$$

Problem 12

(a) $\underset{S \leftarrow \mathcal{B}_1}{P} = \begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix}$

(b) $\underset{\mathcal{B}_1 \leftarrow S}{P} = \dfrac{1}{21-20} \begin{bmatrix} 7 & -5 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 7 & -5 \\ -4 & 3 \end{bmatrix}$

(c) $\left[\begin{array}{cc|cc} 2 & 3 & 3 & 5 \\ -1 & -1 & 4 & 7 \end{array}\right] \longrightarrow \left[\begin{array}{cc|cc} 1 & 0 & -15 & -26 \\ 0 & 1 & 11 & 19 \end{array}\right]$

$\Rightarrow \underset{\mathcal{B}_2 \leftarrow \mathcal{B}_1}{P}$

(d) $\underset{\mathcal{B}_1 \leftarrow \mathcal{B}_2}{P} = \left(\underset{\mathcal{B}_2 \leftarrow \mathcal{B}_1}{P}\right)^{-1} = \dfrac{1}{-285+286} \begin{bmatrix} 19 & 26 \\ 11 & -15 \end{bmatrix} = \begin{bmatrix} 19 & 26 \\ 11 & -15 \end{bmatrix}$

(e) By inspection, $(8,11) = 1 \cdot (3,4) + 1 \cdot (5,7)$

$\Rightarrow [(8,11)]_{\mathcal{B}_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(f) $[(8,11)]_{\mathcal{B}_2} = \underset{\mathcal{B}_2 \leftarrow \mathcal{B}_1}{P} [(8,11)]_{\mathcal{B}_1}$

$= \begin{bmatrix} -15 & -26 \\ 11 & 19 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} -41 \\ 30 \end{bmatrix}$

Problem 13

$\underset{\mathcal{B}_2 \leftarrow \mathcal{B}_1}{P} = \Big[[1]_{\mathcal{B}_2} \; [x]_{\mathcal{B}_2} \; [1+x+x^2]_{\mathcal{B}_2} \Big] = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$[1]_{\mathcal{B}_2} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$[x]_{\mathcal{B}_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$[1+x+x^2]_{\mathcal{B}_2} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ as $1+x+x^2 = -1 \cdot 1 + 1 \cdot x + 1 \cdot (2+x^2)$ by inspection

$\underset{\mathcal{B}_1 \leftarrow \mathcal{B}_2}{P} = \Big[[1]_{\mathcal{B}_1} \; [x]_{\mathcal{B}_1} \; [2+x^2]_{\mathcal{B}_1} \Big] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

$[1]_{\mathcal{B}_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$[x]_{\mathcal{B}_1} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$[2+x^2]_{\mathcal{B}_1} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ as $2+x^2 = 1 \cdot 1 + (-1) \cdot x + 1 \cdot (1+x+x^2)$