BASIC MATHEMATICAL LOGIC

1. Proposition

A **proposition** is a statement that is either true or false. For example, "4 is divisible by 2" is a *true proposition*, "column vectors always have two real entires" is a *false proposition*, and "apples are delicious" is *not a proposition*. Showing that a proposition is true can be very difficult. Some proposition, Fermat's last theorem for instance, took over 350 years to be proven true.

2. Connective

Let P and Q be two propositions. Then the newly formed proposition "P and Q" is true exactly when P and Q are both true. The proposition "P or Q", on the other hand, is true when either one or both are true.

3. Implication

Let P and Q be two propositions. An **implication** is a proposition of the form "If P, then Q" and we denote them by $P \Rightarrow Q$. Here P is called the **hypothesis** and Q is called the **conclusion**. Implications can be often written differently as "Q if P" or "Q whenever P". A statement $P \Rightarrow Q$ is true whenever P is false or both P and Q are true. An example of an implication would be "If the sky is overcast, then the sun is not visible" assuming that we are at the surface of the earth.

The **converse** of the proposition $P \Rightarrow Q$ is $Q \Rightarrow P$. ⚠ *Warning*: $P \Rightarrow Q$ being true does not mean that its converse $Q \Rightarrow P$ is true! The converse of the example above is "If the sun is not visible, then the sky is overcast" which is not always true as it might be clear night. If both the implication and its converse are true, then we say that P and Q are **logically equivalent**. In this case, P is true exactly when Q is true, and vice versa. Conventionally, we write "P if and only if Q" or "P iff Q".

4. Quantifiers

Mathematical propositions and theorems are usually implications with what is called quantifiers. "For all", "For every", "For each", or "For any" are all phrases called **universal quantifiers** that are used interchangeably. For example, "If $x < 1$, then $x^2 < 1$" is, in fact, "For all $x \in \mathbb{R}$, if $x < 1$, then $x^2 < 1$". Sometimes, it may be redundant to include the quantifier, so instead of "For all $x \in \mathbb{R}$, if $x \in \mathbb{R}$, then $x^2 \geq 0$", we write "If $x \in \mathbb{R}$, then $x^2 \geq 0$". Note that whenever you write "for all", you have to specify the set where x belongs. In the example, x was an element of $\mathbb{R}$ the set Observe how the truth value of the propositions above depend on x, so we write $P(x)$ for such propositions.

The phrases "there exists", "there is", or "for some" are called **existential quantifiers** and are used interchangeably. For example,

$$\text{"there is } \mathbf{x} \in \mathbb{R}^2, \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}\text{"}$$

is a proposition with an existential quantifer.

5. Negation and Contrapositive

Let P be a proposition. Then the **negation** "not P" or $\neg P$ is a mathematical proposition such that P is true iff $\neg P$ is false. The **contrapositive** of "$P \Rightarrow Q$" is "$\neg Q \Rightarrow \neg P$". Unlike the converse, contrapositive is logically equivalent to the original proposition. Without too much explanations, we list all necessary negations. Let X be some set. Here $\equiv$ means logically equivalent.

- $\neg(P \text{ and } Q) \equiv (\neg P) \text{ or } (\neg Q)$.
- $\neg(P \text{ or } Q) \equiv (\neg P) \text{ and } (\neg Q)$.
- $\neg(\text{for all } x \in X, P(x)) \equiv \text{for some } x \in X, \neg P(x)$.
- $\neg(\text{for some } x \in X, P(x)) \equiv \text{for all } x \in X, \neg P(x)$.
- $\neg(P \Rightarrow Q) \equiv P \text{ and } \neg Q$.