Chapter 2. Matrix Algebra

Keywords: Matrix multiplication, transpose of a matrix A^T, inverse matrix A^{-1}

(i) $A\begin{bmatrix} b_1 & \cdots & b_n \end{bmatrix} = \begin{bmatrix} Ab_1 & \cdots & Ab_n \end{bmatrix}$	(ii) $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ for 2×2 A with $ad - bc \neq 0$	(iii) $\begin{bmatrix} A \mid I \end{bmatrix} \to \begin{bmatrix} I \mid A^{-1} \end{bmatrix}$ for any square matrix A
(iv) $(AB)^T = B^T A^T$ for any matrices A, B	(v) $(AB)^{-1} = B^{-1}A^{-1}$ for invertible matrices A, B	(vi) $(A^{-1})^T = (A^T)^{-1}$ for an invertible matrix A

Theorem 1. An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n.

Theorem 2. (Invertible Matrix Theorem) For an $n \times n$ matrix A, (a)-(l) are all equivalent
(i) A has n pivot (columns) $\Leftrightarrow$ (d) $\Leftrightarrow$ (e) $\Leftrightarrow$ (f) (ii) A has n pivot (rows) $\Leftrightarrow$ (g) $\Leftrightarrow$ (h) $\Leftrightarrow$ (i)

Chapter 3. Determinants

Keywords: Determinants, Cofactor Expansion across a row or a column, relationship between row operations and determinants, Cramer's Rule, Areas and volumes as determinants.

Defintion 3. Let A be an $n \times n$-matrix.

(a) The submatrix A_{ij} is an $(n-1) \times (n-1)$-matrix obtained from A by deleting ith row and jth column.

(b) **determinant** of A is recursively defined as

$$a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n}$$

Defintion 4. The (i, j)-cofactor of A is

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

cofactor expansion across row i

$$\det A = a_{i1}C_{i1} + \cdots + a_{in}C_{in}$$

cofactor expansion down column j

$$\det A = a_{1j}C_{1j} + \cdots + a_{nj}C_{nj}$$

Theorem 5. If A is a triangular matrix, then $\det A$ is the product of the entries on the diagonal of A.

Theorem 6. A square matrix A is invertible if and only if $\det A \neq 0$.

Theorem 7. $\det A = \det A^T$ and

$$\det(AB) = (\det A)(\det B)$$

(a) $\det B = \det A$ if B is obtained by adding a multiple of another row.

(b) If B is obtained by interchanging two rows, then $\det B = -\det A$.

(c) If B is obtained by multipying k to a row, $\det B = k \det A$.

Cramer's Rule Let A be invertible $n \times n$-matrix and $b \in \mathbb{R}^n$. Then the ith entry x_i of the unique solution is given by

$$x_i = \frac{\det A_i(b)}{\det A}$$

where $A_i(b) = \begin{bmatrix} a_1 & \cdots & b & \cdots & a_n \end{bmatrix}$.

Let A be an invertible $n \times n$ matrix. Then [(1)]

$$A^{-1} = \frac{1}{\det A} \text{adj } A = \frac{1}{\det A}\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

Determinant and Volumes

(a) (Parallelogram) Let $v_1, v_2 \in \mathbb{R}^2$. Then the area of the parallelogram formed by v_1 and v_2 is $\det A$ where $A = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$.

(b) (Parallelopiped) Let $v_1, v_2, v_3 \in \mathbb{R}^3$. Then the volume of the parallelopiped formed by v_1, v_2, v_3 is $\det A$ where $A = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$.

(a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation with standard matrix A. If S is a region in $\mathbb{R}^2$ with finite area. Then

$$\text{area of } T(S) = |\det A| \cdot \text{area of } S$$

(b) $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation with standard matrix A. If S is a region in $\mathbb{R}^3$ with finite volume. Then

$$\text{volume of } T(S) = |\det A| \cdot \text{volume of } S$$

[(1)] The (i, j)-entry of A^{-1} is C_{ji} divided by $\det A$, NOT C_{ij}.

Keywords: vector space, subspace, basis, null, column, and row spaces, dimension, coordinate vectors, change-of-coordinate matrix, rank-nullity theorem.

Defintion 8. A **vector space** is a nonempty set V of objects called **vectors** with two operations *addition* and *mutlipliaction by scalars* (real numbers) with ten properties in pg. 202-203.

Defintion 9. A **subspace** of a vector space V is a subset W of V such that (i) $0 \in W$, (ii) is closed under addition, and (iii) is closed under scalar multiplication.

Defintion 10. A **linear transformation** $T : V \to W$ is a function such that $T(u+v) = T(u) + T(v)$ and $T(cu) = cT(u)$.

Example 11. The set $\mathbb{R}^n$ of column vectors with n entries and the set $\mathbb{P}_n$ of polynomials of degree at most n are vector spaces.(2) Also, a subspace of a vector space is a vector space.

Example 12. Let $v_1, \ldots, v_p \in V$. Then the **span**

$$\text{Span}\{v_1, \ldots, v_p\} = \{c_1v_1 + \cdots + c_pv_p \mid c_i \in \mathbb{R}\}$$

of $\{v_1, \ldots, v_n\}$ is a subspace of V. This is called the **subspace** of V generated by $\{v_1, \ldots, v_p\}$.

(2) The addition and scalar multiplications of $\mathbb{R}^n$ and $\mathbb{P}_n$ are defined differently.

Defintion 13. A set of vectors $\{v_1, \ldots, v_p\}$ in V is **linearly independent** if the **linear dependence relation** $c_1v_1 + \cdots + c_pv_p = 0$ has only trivial solution.

Theorem 14. $\{v_1, \ldots, v_p\}$ with $v_1 \neq 0$ is linearly dependent if and only if some v_j $(j > 1)$ is a linear combination of $v_1, \ldots, v_{j-1}$.

Defintion 15. A indexed set $\mathscr{B}$ of vectors in a vector space V is called a **basis** if (i) $\mathscr{B}$ is linearly independent and (ii) $\text{Span}\ \mathscr{B} = V$.

Defintion 16. If a vector space V is spanned by a finite set, then V is said to be **finite-dimensional** and the number of vectors in a basis is called a **dimension** of V. (3)

(3) A basis of a vector space is not unique, but they all have the same number of vectors by Theorem 18.

Theorem 17. (Spanning Set Theorem) Let $S = \{v_1, \ldots, v_p\}$ be a subset of V and let $W = \text{Span}\ S$. Then (i) if $v_k \in S$ is a linear combination of the remaining vectors in S, then the set $\{v_1, \ldots, \cancel{v_k}, \ldots, v_p\}$ formed by removing v_k from S still spans W and (ii) if $S \neq \{0\}$, then a subset of S is a basis of W.

Theorem 18. (a) If a vector space V has a basis $\mathscr{B}$ with n vectors, then any set in V containing more than n vectors must be linearly dependent. Also, every basis of V must consist of exactly n vectors. (b) Every vector can be written uniquely as a linear combinations of vectors in $\mathscr{B}$.

Theorem 19. Let W be a subspace of a finite-dimensional vector space. Then $\dim W \le \dim V$.

Theorem 20. For $\dim V = n$, (i) any linearly independent set of V with n-elements or (ii) any spanning set of V with n-elements is a basis.

Let A be a $m \times n$-matrix. Write $A = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix}$ where a_is are the column vectors of A. Also, let $r_1, \ldots, r_m$ be its row vectors. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation defined by A.

Subspace	A basis(4)	Dimension
$\text{Nul}\ A = \{x \in \mathbb{R}^n \mid Ax = 0\}$ $= \text{Ker}(T)$ subspace of $\mathbb{R}^n$	$\mathscr{B}$ is the set of vectors appearing in the general solution in parametric vector form	$\text{nullity}\ A = \dim \text{Nul}\ A$
$\text{Col}\ A = \text{Span}\ \{a_1, \ldots, a_n\}$ $= \text{Range}(T)$ subspace of $\mathbb{R}^m$	$\mathscr{B} = \{$ pivot columns of A $\}$	$\text{rank}\ A = \dim \text{Col}\ A$
$\text{Row}\ A = \text{Span}\ \{r_1, \ldots, r_n\}$ is a subspace of $\mathbb{R}^n$	$\mathscr{B}$ is the set of nonzero row vectors of an echelon form B of A	$\text{Row}\ A = \text{Row}\ B$ $A \to B$ row. eq.

Theorem 21 (Rank-Nullity).

$$\text{rank}\ A + \text{nullity}\ A = \#\text{ of cols. of } A$$

Defintion 22. The **standard basis** is the set $\{e_1, \ldots, e_n\}$ in $\mathbb{R}^n$ where e_i is the vector whose entries are all zero except 1 at the ith entry.

Defintion 23. $\mathscr{B} = \{b_1, \ldots, b_n\}$ be a basis.

$$[x]_{\mathscr{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

given $x = c_1b_1 + \cdots + c_nb_n$, is called **coordinate vector** of x relative to $\mathscr{B}$.

Defintion 24. $\mathscr{B} = \{b_1, \ldots, b_n\}$, $\mathscr{C} = \{c_1, \ldots, c_n\}$ be bases of V.

$$\underset{\mathscr{C} \leftarrow \mathscr{B}}{P} = \begin{bmatrix} [b_1]_{\mathscr{C}} & \cdots & [b_n]_{\mathscr{C}} \end{bmatrix}$$

is called the **change-of-coordinates matrix** from $\mathscr{B}$ to $\mathscr{C}$. Also,

$$\left[c_1 \cdots c_n \middle| b_1 \cdots b_n \right] \to \left[I_n \middle| \underset{\mathscr{C} \leftarrow \mathscr{B}}{P} \right]$$

(4) There are infinitely many basis to a vector space. This is just one of them.