Problem 1.

(a) Since the matrix is (upper) triangular, the eigenvalues are $\lambda = 1, 2, 3$.

(b)
$$A - \lambda = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix}$$
Then
$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$
is an eigenvector of A corresponding to 1 (by observation). To see this without guess-and-check, note that for $\begin{bmatrix} x & y & z \end{bmatrix}^T$ to be an eigenvector for 1, by the first and the third row, we must have $y = 0, z = 0$.

(c) A is diagonalizable because A is 3×3 and has three distinct eigenvalues.

(d) A is invertible because $\det(A) = 1 \cdot 2 \cdot 3 = 6 \neq 0$.

Problem 2. Since $Av = \lambda v$, we have $A^2 v = A(\lambda v) = \lambda A v = \lambda^2 v$. By the same reasoning,
$$0 = A^5 v = \lambda^5 v.$$
For a nonzero vector v, $\lambda^5 v = 0 \Rightarrow \lambda^5 = 0 \Rightarrow \lambda = 0$.

Problem 3. We want to find the least squares solution to the system
$$\underbrace{\begin{bmatrix} 1 & 1 \\ -3 & 3 \\ -2 & 1 \\ 2 & -4 \end{bmatrix}}_{=A} x = \underbrace{\begin{bmatrix} 2 \\ 0 \\ 1 \\ -3 \end{bmatrix}}_{=b}$$
We have
$$A^T A = \begin{bmatrix} 1 & -3 & -2 & 2 \\ 1 & 3 & 1 & -4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -3 & 3 \\ -2 & 1 \\ 2 & -4 \end{bmatrix} = \begin{bmatrix} 18 & -18 \\ -18 & 27 \end{bmatrix}$$
$$A^T b = \begin{bmatrix} 1 & -3 & -2 & 2 \\ 1 & 3 & 1 & -4 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} -6 \\ 15 \end{bmatrix}$$
The augmented matrix corresponding to $A^T A x = A^T b$ is
$$\begin{bmatrix} 18 & -18 & -6 \\ -18 & 27 & 15 \end{bmatrix} \to \begin{bmatrix} 3 & -3 & -1 \\ 0 & 9 & 9 \end{bmatrix}$$
Therefore, $x_2 = 1$, and $x_1 = \frac{2}{3}$. The least square solution is thus,
$$\hat{x} = \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$$

Problem 4. We want to find the least squares solution to
$$\underbrace{\begin{bmatrix} 1 & -1 \\ -2 & 1 \\ 1 & -3 \end{bmatrix}}_{=A} x = \underbrace{\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}}_{=b}$$

Then normalized equation is

$$A^T A = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & -3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 1 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 6 & -6 \\ -6 & 11 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & -3 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$$

The augmented matrix is

$$\begin{bmatrix} 6 & -6 & 6 \\ -6 & 11 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 \\ 0 & 5 & 4 \end{bmatrix}$$

Then $x_2 = \frac{4}{5}$ and $x_1 = 1 + x_2 = \frac{9}{5}$. Therefore, the least square solution is

$$\hat{x} = \begin{bmatrix} \frac{9}{5} \\ \frac{4}{5} \end{bmatrix}$$

Problem 5. The point in W that is closest to the point y is

$$\text{Proj}_W y = \frac{y \cdot (1,1)}{(1,1) \cdot (1,1)}(1,1) = \frac{(3,4) \cdot (1,1)}{(1,1) \cdot (1,1)}(1,1) = \frac{7}{2}(1,1) = \left(\frac{7}{2}, \frac{7}{2}\right).$$

Problem 6. By orthogonal decomposition,

$$y = \text{Proj}_W y + \underbrace{z}_{\in W^T} = \left(\frac{7}{2}, \frac{7}{2}\right) + z \Rightarrow z = \left(-\frac{1}{2}, \frac{1}{2}\right)$$

To conclude,

$$y = \underbrace{\left(\frac{7}{2}, \frac{7}{2}\right)}_{\in W} + \underbrace{\left(-\frac{1}{2}, \frac{1}{2}\right)}_{\in W^T}.$$

Problem 7.

$$\begin{aligned} w' = \text{Proj}_W y &= \frac{(-5,2,2) \cdot (-2,3,3)}{(-2,3,3) \cdot (-2,3,3)}(-2,3,3) + \frac{(-5,2,2) \cdot (-3,0,-2)}{(-3,0,-2) \cdot (-3,0,-2)}(-3,0,-2) \\ &\Rightarrow \frac{22}{22}(-2,3,3) + \frac{11}{13}(-3,0,-2) \\ &= \left(-\frac{26}{13}, 3, \frac{39}{13}\right) + \left(-\frac{33}{13}, 0, -\frac{22}{13}\right) = \left(-\frac{59}{13}, 3, \frac{17}{13}\right) \end{aligned}$$

Solving for z, we should see that

$$(-5,2,2) = \left(-\frac{59}{13}, 3, \frac{17}{13}\right) + z = \left(-\frac{59}{13}, 3, \frac{17}{13}\right) + \left(-\frac{6}{13}, -1, \frac{9}{13}\right)$$

Problem 8. Let $x_1 = (1,0,1)$ and $x_2 = (0,1,1)$. Then the Gram-Schmidt process yields

$$\begin{aligned} v_1 &= x_1 = (1,0,1) \\ v_2 &= x_2 - \tfrac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1 = (0,1,1) - \tfrac{(0,1,1)\cdot(1,0,1)}{(1,0,1)\cdot(1,0,1)}(1,0,1) \\ &= (0,1,1) - \tfrac{1}{2}(1,0,1) = (-\tfrac{1}{2}, 1, \tfrac{1}{2}) \end{aligned}$$

Then $\{(1,0,1), (-\frac{1}{2}, 1, \frac{1}{2})\}$ is an orthogonal basis. Next,

$$\begin{aligned} u_1 &= \tfrac{v_1}{||v_1||} = \left(\tfrac{1}{\sqrt{2}}, 0, \tfrac{1}{\sqrt{2}}\right) \\ u_2 &= \tfrac{v_2}{||v_2||} = \tfrac{\sqrt{2}}{\sqrt{3}}(-\tfrac{1}{2}, 1, \tfrac{1}{2}) = \left(-\tfrac{1}{\sqrt{6}}, \tfrac{2}{\sqrt{6}}, \tfrac{1}{\sqrt{6}}\right) \end{aligned}$$

Then $\left\{\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)\right\}$ is an orthonormal basis.

Problem 9.

(a) **False** Let v be an eigenvector corresponding to λ. Then av is also an eigenvector corresponding to λ for any $a \neq 0$. If you have one, then you have infinitely many of them.

(b) **True** An orthogonal set of nonzero vectors is automatically linearly independent. You cannot have linearly independent set in $\mathbb{R}^5$ with 6 elements because the dimension of $\mathbb{R}^5$ is 5.

(c) **False**. Consider the following least squares problem

$$\underbrace{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}}_{=A} x = \underbrace{\begin{bmatrix}0\\0\end{bmatrix}}_{=b} \quad \Rightarrow \quad \underbrace{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}}_{A^TA} x = \underbrace{\begin{bmatrix}0\\0\end{bmatrix}}_{A^Tb}$$

(d) **True** We use the theorem that says $y \in W$ if and only if $y = \mathrm{Proj}_W y$.

Problem 10.

(a) The matrix for T relative to basis $\mathscr{B}_1$ and $\mathscr{B}_2$ is given by

$$[T]_{\mathscr{B}_2,\mathscr{B}_1} = \big[[T(2,4)]_{\mathscr{B}_2} \quad [T(1,7)]_{\mathscr{B}_2}\big] = \big[[(-2,-2)]_{\mathscr{B}_2} \quad [(5,-1)]_{\mathscr{B}_2}\big]$$

This matrix is constructed in a way that

(†) $$[T]_{\mathscr{B}_2,\mathscr{B}_1}[x]_{\mathscr{B}_1} = [T(x)]_{\mathscr{B}_2}$$

Since

$$\begin{aligned}(-2,-2) &= (1,3) + (-3,-5) &&\Rightarrow [(-2,-2)]_{\mathscr{B}_2} = \begin{bmatrix}1\\1\end{bmatrix}\\ (5,-1) &= -7(1,3) - 4(-3,-5) &&\Rightarrow [(5,-1)]_{\mathscr{B}_2} = \begin{bmatrix}-7\\-4\end{bmatrix}\end{aligned}$$

For the second equation, we can work with the augmented matrix

$$\begin{bmatrix}1 & -3 & 5\\3 & -5 & -1\end{bmatrix}$$

Therefore,

$$[T]_{\mathscr{B}_2,\mathscr{B}_1} = \begin{bmatrix}1 & -7\\1 & -4\end{bmatrix}$$

(b) By observation, $(3,11) = (2,4) + (1,7)$. Therefore, $[(3,11)]_{\mathscr{B}_1} = \begin{bmatrix}1\\1\end{bmatrix}$.

(c) By (a), (b), and (†) above, we have

$$[T(3,11)]_{\mathscr{B}_2} = [T]_{\mathscr{B}_2,\mathscr{B}_1}[(3,11)]_{\mathscr{B}_1} = \begin{bmatrix}1 & -7\\1 & -4\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}-6\\-4\end{bmatrix}.$$

In particular, $T(3,11) = -6(1,3) - 4(-3,-5) = (6,2)$

Problem 11.

(a) Let

$$B = \begin{bmatrix}3 & -4\\-2 & 3\end{bmatrix} \text{ then } \det B = 1 \text{ and } B^{-1} = \begin{bmatrix}3 & 4\\2 & 3\end{bmatrix}$$

Then by Theorem 8 of Chapter 5 (or Theorem 9 of the Study Guide),

$$[T]_{\mathscr{B}} = B^{-1}AB = \begin{bmatrix}3 & 4\\2 & 3\end{bmatrix}\begin{bmatrix}2 & 1\\2 & 3\end{bmatrix}\begin{bmatrix}3 & -4\\-2 & 3\end{bmatrix} = \begin{bmatrix}12 & -11\\8 & -7\end{bmatrix}$$

(b) By observation, $(7,-5) = (3,-2) - (-4,3)$, so $[(7,-5)]_{\mathscr{B}} = \begin{bmatrix}1\\-1\end{bmatrix}$.

(c) We have

$$[T(7,-5)]_{\mathscr{B}} = [T]_{\mathscr{B}}[(7,-5)]_{\mathscr{B}} = \begin{bmatrix} 12 & -11 \\ 8 & -7 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 23 \\ 15 \end{bmatrix}$$

Therefore,

$$T(7,-5) = 23(3,-2) + 15(-4,3) = (9,-1)$$

Problem 12. The characteristic polynomial is

$$\det \begin{bmatrix} 1-\lambda & -5 \\ 1 & -3-\lambda \end{bmatrix} = (1-\lambda)(-3-\lambda) + 5 = \lambda^2 + 2\lambda + 2.$$

Then

$$\lambda = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i.$$

Then

$$A - (-1+i)I = \begin{bmatrix} 2-i & -5 \\ 1 & -2-i \end{bmatrix} \to \begin{bmatrix} 2-i & -5 \\ 0 & 0 \end{bmatrix}$$

Here we used the fact that $(2-i) \cdot \begin{bmatrix} 1 & -2-i \end{bmatrix} = \begin{bmatrix} 2-i & -5 \end{bmatrix}$. Alternatively, we know that the algebraic multiplicty of $-1+i$ is 1, so the dimension of the null space $\mathrm{Null}(A - (-1+i)I) = 1$. Therefore, the row reduction is immediate. Then

$$\begin{bmatrix} 5 \\ 2-i \end{bmatrix}$$

is an eigenvector for $-1+i$. Then we immediately get that an eigenvector for $-1-i$ is

$$\overline{\begin{bmatrix} 5 \\ 2-i \end{bmatrix}} = \begin{bmatrix} 5 \\ 2+i \end{bmatrix}$$

We did not cover the decomposition $A = PCP^{-1}$, so we will skip the solution.

Problem 13. Consider the 3×3 identity matrix $A = I_3$. Then

$$A = I_3 I_3 I_3^{-1}$$

therefore, A is diagonalizable. Since I_3 is an upper triangular matrix, the only eigenvalue is 1.

Problem 14. The characteristic polynomial is

$$\det \begin{bmatrix} 1-\lambda & 5 \\ 5 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 - 25 = \lambda^2 - 2\lambda - 24 = (\lambda - 6)(\lambda + 4)$$

Therefore, the eigenvalues are $\lambda = 6, -4$.

$$A - 6I = \begin{bmatrix} -5 & 5 \\ 5 & -5 \end{bmatrix} \to \begin{bmatrix} -5 & 5 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$$

so, an eigenvector of A for 6 is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$A + 4I = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

so, an eigenvector of A for 4 is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Therefore, $P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Since $\det P = -2$, we have

$$P^{-1} = -\frac{1}{2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

Therefore, the diagonalization is

$$\begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 6 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$