CLOSED FORMULA OF THE FIBONACCI SEQUENCE

The Fibonacci sequence is defined recursively as

$$a_0 = 0, a_1 = 1, \quad a_2 = 1, \quad \text{and } a_n = a_{n-1} + a_{n-2} \text{ for } n \geq 2$$

Therefore, $a_3 = a_2 + a_1 = 2$, $a_4 = a_3 + a_2 = 3$, and so on. As of now, one must compute all $a_1, \ldots, a_{n-1}$ to compute a_n. However, with help of diagonalization of matrices, one can find a closed formula for Fibonacci sequence. In order to to land in the realm of linear algebra, we wish to compute a_n via some matrix multiplication. To do this, let us define a vector

$$v_n = \begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix}$$

for $n \geq 0$. Then we want to find a 2×2-matrix A such that

$$v_{n+1} = Av_n.$$

Indeed,

$$v_{n+1} = \begin{bmatrix} a_{n+1} \\ a_{n+2} \end{bmatrix} = \begin{bmatrix} a_{n+1} \\ a_{n+1} + a_n \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix} \quad \Rightarrow \quad A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

Then we get the equality

$$v_n = A^n v_0 = A^n \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \text{second column of } A^n.$$

In particular, the $(1, 2)$-entry of A^n is equal to a_n. Hence our goal is to efficiently compute A^n. This is where diagaonalization comes in handy. To diagonalize A, we start by computing its characteristic polynomial of A,

$$\det \begin{bmatrix} -\lambda & 1 \\ 1 & 1 - \lambda \end{bmatrix} = \lambda^2 - \lambda - 1.$$

By the quadratic formula, we see that the eigenvalues are

$$\lambda = \frac{1 \pm \sqrt{5}}{2}.$$

which are distinct. Since A is 2×2-matrix, it follows that A is diagonalizable. We denote by $\lambda_1 = \frac{1-\sqrt{5}}{2}$ and $\lambda_2 = \frac{1+\sqrt{5}}{2}$. It is useful to observe that

$$\lambda_1 + \lambda_2 = 1 \text{ and } \lambda_1 \lambda_2 = -1$$

as well as $1 - \lambda_1 = \lambda_2$ and $1 - \lambda_2 = \lambda_1$.

Next we find eigenvectors for λ_1 and λ_2.

$$A - \lambda_1 I = \begin{bmatrix} -\lambda_1 & 1 \\ 1 & 1 - \lambda_1 \end{bmatrix} = \begin{bmatrix} -\lambda_1 & 1 \\ 1 & \lambda_2 \end{bmatrix}$$

Observe that

$$\lambda_2 \begin{bmatrix} -\lambda_1 & 1 \end{bmatrix} = \begin{bmatrix} -\lambda_1 \lambda_2 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 & \lambda_2 \end{bmatrix}$$

Therefore,

$$A - \lambda_1 I = \begin{bmatrix} \lambda_2 & 1 \\ 1 & -\lambda_1 \end{bmatrix} \to \begin{bmatrix} -\lambda_1 & 1 \\ 0 & 0 \end{bmatrix}$$

We thus can conclude that

$$\begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} \quad \text{is an eigenvector for } \lambda_1$$

$$\begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix} \quad \text{is an eigenvector for } \lambda_2 \quad \text{by the same process after interchanging } \lambda_1 \text{ and } \lambda_2$$

Let P be the matrix formed by these eigenvectors, i.e.

$$P = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix}.$$

Then

$$\det(P) = \lambda_2 - \lambda_1 = \frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2} = \sqrt{5}.$$

In particular,

$$P^{-1} = \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{bmatrix}$$

Then the diagonalization of A is given by

$$A = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{bmatrix}$$

The nth-power of A is

$$A^n = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_1^n & \lambda_2^n \\ \lambda_1^{n+1} & \lambda_2^{n+1} \end{bmatrix} \begin{bmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{bmatrix}$$

Then the $(1,2)$-entry of A^n is

$$a_n = \frac{1}{\sqrt{5}}(\lambda_2^n - \lambda_1^n) = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right).$$