Worksheet 1	**Name:**	______________________
MATH 240 (Spring 2024)	**Section:**	0111(12PM-1PM) / 0121(1PM-2PM)
Jan 30, 2024	**TA:**	Shin Eui Song

1. **Warm-up Problems**

Problem 1. Determine if the following linear systems has no solution, one solution, and infinite solutions. If the system has one solution, find it.

a)

$$\begin{aligned} 3x_1 + 2x_2 &= 7 \\ -9x_1 - 6x_2 &= -21 \end{aligned}$$

b)

$$\begin{aligned} 3x_1 + 2x_2 &= 7 \\ -2x_1 + 3x_2 &= 4 \end{aligned}$$

c)

$$\begin{aligned} x_1 - 2x_2 &= 2 \\ -2x_1 + 4x_2 &= 1 \end{aligned}$$

2. **Suggested Problems**

Problem 2 (1.1.14)**.** Solve the following system

$$\begin{aligned} x_1 - 3x_2 &= 5 \quad (1) \\ -x_1 + x_2 + 5x_3 &= 2 \quad (2) \\ x_2 + x_3 &= 0 \quad (3) \end{aligned}$$

Problem 3 (1.1.19)**.** Determine if the system is consistent. Do not completely solve the system.

$$\begin{aligned} x_1 + 3x_3 &= 2 \quad (1) \\ x_2 - 3x_4 &= 3 \quad (2) \\ -2x_2 + 3x_3 + 2x_4 &= 1 \quad (3) \\ 3x_1 + 7x_4 &= -5 \quad (4) \end{aligned}$$

Problem 4 (1.1.22)**.** Do the three planes $x_1 + 2x_2 + x_3 = 4$, $x_2 - x_3 = 1$, and $x_1 + 3x^2 = 0$ have at least one common point of intersection? Explain.

3. **Additional Problems**

Problem 5 (Challenge)**.** Consider the linear system

$$\begin{aligned} ax_1 + 2x_2 &= 1 \\ x_1 - 3x_2 &= -10 \end{aligned}$$

(a) Can you find a value for a such that the linear system is inconsistent? If not, explain why.

(b) Can you find a value for a such that the linear system has infinite solutions? If not, explain why.

(c) Find an **integer** value for a such that the linear system have one solution and x_1 is a **negative integer**? Is such a unique?

Worksheet 1 Solution
MATH 240 (Spring 2024)
Jan 30, 2024

Problem 1.

(a) Consider

$$\begin{bmatrix} 3 & 2 & 7 \\ -9 & -6 & -21 \end{bmatrix} \xrightarrow{R_2=R_2+3R_1} \begin{bmatrix} 3 & 2 & 7 \\ 0 & 0 & 0 \end{bmatrix}$$

Then the solution set of this matrix is the solution set of the linear equation $3x_1 + 2x_2 = 7$. For any x_1, we can always find $x_2 = (7 - 3x_1)/2$. Therefore, the linear system has infinitely many solutions.

(b) Consider

$$\begin{bmatrix} 3 & 2 & 7 \\ -2 & 3 & 4 \end{bmatrix} \xrightarrow{2R_1} \begin{bmatrix} 6 & 4 & 14 \\ -2 & 3 & 4 \end{bmatrix} \xrightarrow{3R_2} \begin{bmatrix} 6 & 4 & 14 \\ -6 & 9 & 12 \end{bmatrix} \xrightarrow{R_2+R_1} \begin{bmatrix} 6 & 4 & 14 \\ 0 & 13 & 26 \end{bmatrix} \xrightarrow{\frac{1}{13}R_2} \begin{bmatrix} 6 & 4 & 14 \\ 0 & 1 & 2 \end{bmatrix}$$

Then $x_2 = 2$. Plugging back to the first linear equation $3x_1 + 2x_2 = 7$, we get $x_1 = 1$. There is only one solution with $x_1 = 1$ and $x_2 = 2$.

(c)

$$\begin{bmatrix} 1 & -2 & 2 \\ -2 & 4 & 1 \end{bmatrix} \xrightarrow{R_2=R_2+2R_1} \begin{bmatrix} 1 & -2 & 2 \\ 0 & 0 & 5 \end{bmatrix}$$

However, there is no x_1 and x_2 such that $0x_1 + 0x_2 = 5$. Therefore, the linear equation has no solution.

Problem 2. When the coefficient matrix can be row reduced to a *triangular form* (later called the echelon form), we can immediately say that the linear system is consistent.

$$\begin{bmatrix} 1 & -3 & 0 & 5 \\ -1 & 1 & 5 & 2 \\ 0 & 1 & 1 & 0 \end{bmatrix} \xrightarrow{R_2+R_1} \begin{bmatrix} 1 & -3 & 0 & 5 \\ 0 & -2 & 5 & 7 \\ 0 & 1 & 1 & 0 \end{bmatrix} \xrightarrow{R_2+2R_3} \begin{bmatrix} 1 & -3 & 0 & 5 \\ 0 & 0 & 7 & 7 \\ 0 & 1 & 1 & 0 \end{bmatrix} \xrightarrow{R_2\leftrightarrow R_3} \begin{bmatrix} 1 & -3 & 0 & 5 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 7 & 7 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{7}\cdot R_3} \begin{bmatrix} 1 & -3 & 0 & 5 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_2-R_3} \begin{bmatrix} 1 & -3 & 0 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_1+3R_2} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Here the blue matrix (the coefficient matrix) is of the *triangular form*. After row reducing to the triangular form, we can continue the algorithm of solving linear system to reach to the last matrix. Then we immediately get $x_1 = 2$, $x_2 = -1$, and $x_3 = 1$.

Problem 3. Consider

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & -2 & 3 & 2 & 1 \\ 3 & 0 & 0 & 7 & -5 \end{bmatrix} \xrightarrow{R_4-3R_1} \begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & -2 & 3 & 2 & 1 \\ 0 & 0 & -9 & 7 & -11 \end{bmatrix} \xrightarrow{R_3+2R_2} \begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & 0 & 3 & -4 & 7 \\ 0 & 0 & -9 & 7 & -11 \end{bmatrix}$$

$$\xrightarrow{R_4+3R_3} \begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & 0 & 3 & -4 & 7 \\ 0 & 0 & 0 & -5 & 10 \end{bmatrix}$$

Then we found that the coefficient matrix can be row reduced to a triangular form. This means that the linear system is consistent.

Problem 4. Consider

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 1 & 3 & 0 & 0 \end{bmatrix} \xrightarrow{R_3 - R_1} \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -4 \end{bmatrix}$$

Then row 2 and row 3 of the last matrix yields a linear system

$$\begin{aligned} x_2 - x_3 &= 1 \\ x_2 - x_3 &= -4 \end{aligned}$$

that clearly does not have a solution. The system is inconsistent.

Problem 5. Consider

$$\begin{bmatrix} a & 2 & 1 \\ 1 & -3 & -10 \end{bmatrix} \xrightarrow{3R_1} \begin{bmatrix} 3a & 6 & 3 \\ 1 & -3 & -10 \end{bmatrix} \xrightarrow{2R_2} \begin{bmatrix} 3a & 6 & 3 \\ 2 & -6 & -20 \end{bmatrix} \xrightarrow{R_2 + R_1} \begin{bmatrix} 3a & 6 & 3 \\ 3a + 2 & 0 & -17 \end{bmatrix}$$

(a) If $3a + 2 = 0$, i.e. $a = -\frac{2}{3}$ the second row becomes $\begin{bmatrix} 0 & 0 & -17 \end{bmatrix}$. Hence there is no solution, i.e. inconsistent.

(b) If $3a + 2 \neq 0$, then $x_1 = \frac{-17}{3a+2}$. Furthermore,

$$3ax_1 + 6x_2 = 3 \Rightarrow x_2 = \frac{3 - 3ax_1}{6}$$

So whenever x_1 is defined, x_2 is defined. Therefore, the linear system always has only one solution. To conclude, the linear system cannot have infinite solutions.

(c) We have that $x_1 = \frac{-17}{3a+2}$. Since 17 is a prime number, the only way x_1 is a negative integer, $3a + 2 = 1$ or 17. In particular, a is not unique.