Worksheet 10	**Name:**	______________________________
MATH 240 (Spring 2024)	**Section:**	0111(12PM-1PM) / 0121(1PM-2PM)
Feb 29, 2024	**TA:**	Shin Eui Song

1. Suggested Problems

Problem 1 (2.2.34)**.** Suppose A is $n \times n$ and the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbb{R}^n$. Explain why A must be invertible without using Theorem 8. [Hint: Is A row equivalent to I_n?]

Problem 2 (2.2.41)**.** Find the inverses of the matrix by using the algorithm introduced in this section (find the echelon form of $\begin{bmatrix} A & I_3 \end{bmatrix}$).

$$A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$$

Problem 3 (2.3.5)**.** Using as few calculation as possible, determine if the following matrix is invertible. Justify your answer.

$$\begin{bmatrix} 0 & 3 & -5 \\ 1 & 0 & 2 \\ -4 & -9 & 7 \end{bmatrix}$$

Problem 4 (2.3.23)**.** Can a square matrix with two identical columns be invertible? Why or why not?

Problem 5 (2.3.29)**.** If the equation $G\mathbf{x} = \mathbf{y}$ has more than one solution for some $\mathbf{y}$ in $\mathbb{R}^n$, can the columns of G span $\mathbb{R}^n$? Why or why not?

Problem 6 (2.2.25)**.** Suppose A and B are $n \times n$, B is invertible, and AB is invertible. Show that A is invertible.

Problem 7 (2.3.35)**.** Show that if AB is invertible, so is A. You cannot use Theorem 6(b), because you cannot assume that A and B are invertible. [Hint: There is a matrix W such that $ABW = I$.]

2. Additional Problems

By Theorem 8, we see that if $AB = I_n$ if and only if $BA = I_n$. This works because we are dealing with *finite* matrices. The following problem shows $AB = I$ may not be same as $BA = I$ for *infinite matrices*. We have used the fact that

$$\{ \text{ matrices } \} \leftrightarrow \{ \text{ linear transformations } \}$$

Problem 8. Let P denote set of all polynomials of real coefficients with variable x. Let $\mathscr{L}(P)$ be the set of all *linear transformations* of P to P. Namely, it is a fuction $T : P \to P$ such that

i) $T(f + g) = T(f) + T(g)$,

ii) $T(cf) = cT(f)$

for all $f, g \in P$ and $c \in \mathbb{R}$. From Calculus, we know that the derivative operator $D(f) = \frac{d}{dx}f$ and the integration operator $I(f) = \int_0^x f(t)\, dt$ is in $\mathscr{L}(P)$. Let $\mathrm{Id} : P \to P$ be the identity operator $\mathrm{Id}(f) = f$. Prove that $ID = \mathrm{Id}$ but $DI \neq \mathrm{Id}$.

Problem 1. If $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbb{R}^n$, the linear transformation associated to A is onto, so there is pivot positions in every row. Then its reduced echelon form must have 1 on every diagonal and 0 everywhere else. This is exactly I_n. Since A is row equvalent to I_n, it is invertible.

Problem 2.

$$\begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_3=R_3-2R_1]{R_2=R_2+3R_1} \begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & -3 & 8 & -2 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3=R_3+3R_2} \begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{bmatrix} \xrightarrow[R_2=R_2+R_3]{R_1=R_1+R_3} \begin{bmatrix} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3=\frac{1}{2}R_3} \begin{bmatrix} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 1 & \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{bmatrix}$$

Problem 3.

$$\begin{bmatrix} 0 & 3 & -5 \\ 1 & 0 & 2 \\ -4 & -9 & 7 \end{bmatrix} \xrightarrow{R_3=R_3+4R_2} \begin{bmatrix} 0 & 3 & -5 \\ 1 & 0 & 2 \\ 0 & -9 & 15 \end{bmatrix}$$

If $\mathbf{b}_i$ denote the ith column vector, we have $\mathbf{b}_3 = -\frac{5}{3}\mathbf{b}_2$, i.e. they are scalar multiple of each other. Therefore the set of columns of A is not lineraly independent. Therefore, A is not one-to-one, hence not invertible.

Problem 4. If a square matrix has two identical columns, then the column vectors are not linearly independent. Therefore, the associated linear transformation is not one-to-one. This implies that the matrix is not invertible.

Problem 5. If $G\mathbf{x} = \mathbf{y}$ has more than one solution for some $\mathbf{y}$ in $\mathbb{R}^n$, then it cannot be invertible. This means that the linear transformation associated to G is not onto, and this implies that columns of G does not span $\mathbb{R}^n$.

Problem 6. We need to find the inverse of B. We start with $AB(AB)^{-1} = I_n$. Since A is invertible, we see that $B(AB)^{-1} = A^{-1} \Rightarrow B(AB)^{-1}A = I_n$. Also, $(AB)^{-1}AB = I_n$. So $C = (AB)^{-1}A$ is the inverse of B, so B is invertible.

Problem 7. This is a simple question once you understand the set up. Since both D and I are linear, we first show that $DI(a_nx^n) = a_nx^n$.

$$DI(a_nx^n) = D\left(\frac{a_n}{n+1}x^{n+1}\right) = \frac{a_n}{n+1}(n+1)x^n = a_nx_n$$

then

$$DI(f) = DI(a_0 + \cdots a_nx^n) = DI(a_0) + \cdots + DI(a_nx^n) = a_0 + \cdots + a_nx^n = f = \mathrm{Id}(f)$$

On the other hand, consider $f = 1$. Then $ID(1) = I(0) = \int_0^x 0\, dt = 0 \neq \mathrm{Id}(1)$.