Worksheet 11	**Name:** ________________
MATH 240 (Spring 2024)	**Section:** 0111(12PM-1PM) / 0121(1PM-2PM)
Mar 5th, 2024	**TA:** Shin Eui Song

1. Suggested Problems

Problem 1 (3.1.9)**.** Compute the determinants by cofactor expansions. At each step, choose a row or column that involves the least amount of computation.

$$\begin{vmatrix} 4 & 0 & 0 & 5 \\ 1 & 7 & 2 & -5 \\ 3 & 0 & 0 & 0 \\ 8 & 3 & 1 & 7 \end{vmatrix}$$

Problem 2 (3.1.38)**.** Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and let k be a scalar. Find a formula that relates $\det kA$ to k and $\det A$.

Problem 3 (3.2.21)**.** Use determinants to find out if the matrix is invertible.

$$\begin{bmatrix} 2 & 6 & 0 \\ 1 & 3 & 2 \\ 3 & 9 & 2 \end{bmatrix}$$

Problem 4 (3.2.29)**. (T/F)** If the columns of A are linearly dependent, then $\det A = 0$.

Problem 5 (3.2.32)**. (T/F)** The determinant of A is the product of the pivots in any echelon form U of A, multiplied by $(-1)^r$, where r is the number of row interchanges made during row reduction form A to U.

Problem 6 (3.2.45)**.** Let A and B be 3×3 matrices, with $\det A = -2$ and $\det B = 3$. Use properties of determinants (in the text) to compute:

a) $\det AB$
b) $\det 5A$
c) $\det B^T$
d) $\det A^{-1}$
e) $\det A^3$

2. Additional Problem

The definition of determinant may seem unnatural, but it actually arise from *nice* properties of (bi)linear transformations. To do this, we consider matrices as tuples of column vectors. For instance, 2×2 matrix A is simply a pair $(\mathbf{a}_1, \mathbf{a}_2)$ of vectors in $\mathbb{R}^2$.

Problem 7. Let $T : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be a map that inputs a pair of vectors in $\mathbb{R}^2$ and outputs a scalar $\mathbb{R}$. Suppose

(a) T is **bilinear**, i.e.

$T(a\mathbf{x}_1 + b\mathbf{x}_2, \mathbf{y}) = aT(\mathbf{x}_1, \mathbf{y}) + bT(\mathbf{x}_2, \mathbf{y})$ and $T(\mathbf{x}, a\mathbf{y}_1 + b\mathbf{y}_2) = aT(\mathbf{x}, \mathbf{y}_1) + bT(\mathbf{x}, \mathbf{y}_2)$

(b) T is **alternating**, i.e.

$$T(\mathbf{x}, \mathbf{x}) = 0$$

for all $\mathbf{x} \in \mathbb{R}^2$

(c) $T(\mathbf{e}_1, \mathbf{e}_2) = 1$.

We can think of T as a map from the set $M_{2\times2}(\mathbb{R})$ of 2×2 matrices as follows: write $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix}$ where $\mathbf{a}_i$ are column vectors of A. Then $T(A) := T(\mathbf{a}_1, \mathbf{a}_2)$.

(i) Prove that $T(\mathbf{x}, \mathbf{y}) = -T(\mathbf{y}, \mathbf{x})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$. (Hint: $T(\mathbf{x} + \mathbf{y}, \mathbf{x} + \mathbf{y}) = 0$)

(ii) Prove that $T(A) = \det(A)$.

Problem 1. We choose row 3 for our cofactor expansion. Then

$$\begin{vmatrix} 4 & 0 & 0 & 5 \\ 1 & 7 & 2 & -5 \\ 3 & 0 & 0 & 0 \\ 8 & 3 & 1 & 7 \end{vmatrix} = \cancel{(-1)^{3+3}} \cdot 3 \cdot \begin{vmatrix} 0 & 0 & 5 \\ 7 & 2 & -5 \\ 3 & 1 & 7 \end{vmatrix}$$

Next we choose row 1, then

$$\begin{vmatrix} 0 & 0 & 5 \\ 7 & 2 & -5 \\ 3 & 1 & 7 \end{vmatrix} = \cancel{(-1)^{1+3}} \cdot 5 \cdot \begin{vmatrix} 7 & 2 \\ 3 & 1 \end{vmatrix} = 5 \cdot (7-6) = 5$$

To conclude, we have

$$\begin{vmatrix} 4 & 0 & 0 & 5 \\ 1 & 7 & 2 & -5 \\ 3 & 0 & 0 & 0 \\ 8 & 3 & 1 & 7 \end{vmatrix} = 3 \cdot 5 = 15$$

Problem 2. We have $\det(A) = ad - bc$.

$$\det(kA) = \begin{vmatrix} ka & kb \\ kc & kd \end{vmatrix} = k^2ad - k^2bc = k^2(ad - bc) = k^2\det(A)$$

Problem 3. We have

$$\begin{vmatrix} 2 & 6 & 0 \\ 1 & 3 & 2 \\ 3 & 9 & 2 \end{vmatrix} = 2 \cdot (6-18) - 6 \cdot (2-6) + 0 \cdot (9-9) = -24 + 24 = 0$$

Therefore the matrix is not invertible.

Problem 4. True

Columns of A is linearly dependent $\Rightarrow$ A is not one-to-one $\Rightarrow$ A is not invertible $\Rightarrow$ $\det(A) = 0$.

Problem 5. False

Consider

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Since the matrix is already in echelon form,

$$(-1)^r \cdot (\text{product of pivots}) = (-1)^0 \cdot 1 = 1$$

However, $\det(A) = 0$. The formula is true only for invertible matrices, i.e. matrices with $\det(A) \neq 0$.

Problem 6.

(a) $\det(AB) = \det(A)\det(B) = -6$,

(b) $\det(5A) = 5^3\det(A) = -250$,

(c) $\det(B^T) = \det(B) = 3$,

(d) Since $\det(A) \neq 0$, A is invertible. Then $\det(A^{-1}) = \frac{1}{\det(A)} = -\frac{1}{2}$.

(e) $\det(A^3) = \det(A)^3 = -8$.

Problem 7.

(i) we have

$$0 = T(\mathbf{x}+\mathbf{y},\mathbf{x}+\mathbf{y}) = T(\mathbf{x},\mathbf{x}+\mathbf{y}) + T(\mathbf{y}+\mathbf{x}+\mathbf{y}) = T(\mathbf{x},\mathbf{x}) + T(\mathbf{x},\mathbf{y}) + T(\mathbf{y},\mathbf{x}) + T(\mathbf{y},\mathbf{y})$$

Hence

$$T(\mathbf{x},\mathbf{y}) = -T(\mathbf{y},\mathbf{x})$$

(ii) Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix}$$

If $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, we have $\mathbf{a}_1 = a\mathbf{e}_1 + c\mathbf{e}_2$ and $\mathbf{a}_2 = b\mathbf{e}_1 + c\mathbf{e}_2$. Then

$$\begin{aligned}
T(\mathbf{a}_1,\mathbf{a}_2) &= T(a\mathbf{e}_1 + c\mathbf{e}_2, b\mathbf{e}_1 + d\mathbf{e}_2) = aT(\mathbf{e}_1, b\mathbf{e}_1 + d\mathbf{e}_2) + cT(\mathbf{e}_2, b\mathbf{e}_1) + d\mathbf{e}_2) \\
&= \cancelto{0}{abT(\mathbf{e}_1,\mathbf{e}_1)} + adT(\mathbf{e}_1,\mathbf{e}_2) + bcT(\mathbf{e}_2,\mathbf{e}_1) + \cancelto{0}{cdT(\mathbf{e}_2,\mathbf{e}_2)} \\
&= adT(\mathbf{e}_1,\mathbf{e}_2) + bcT(\mathbf{e}_2,\mathbf{e}_1) \\
&= ad - bc = \det(A)
\end{aligned}$$