Worksheet 12	**Name:**	______________________________
MATH 240 (Spring 2024)	**Section:**	0111(12PM-1PM) / 0121(1PM-2PM)
Mar 7th, 2024	**TA:**	Shin Eui Song

1. Suggested Problems

Problem 1 (3.3.5)**.** Use Cramer's rule to compute the solutions of the following system

$$\begin{aligned} x_1 + x_2 \quad\quad\quad &= 3 \\ -3x_1 \quad\quad + 2x_3 &= 0 \\ x_2 - 2x_3 &= 2 \end{aligned}$$

Problem 2 (3.3.23)**.** Find the volume of the parallelpipied with one vertex at the origin and adjacent vertices at $(1, 0, -3)$, $(1, 2, 4)$, and $(5, 1, 0)$.

Problem 3 (3.3.27)**.** Let S be the parallelogram determined by the vectors $\mathbf{b}_1 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$, and let $A = \begin{bmatrix} 6 & -3 \\ -3 & 2 \end{bmatrix}$. Compute the area of the image of S under the mapping $\mathbf{x} \mapsto A\mathbf{x}$.

Problem 4 (3.S.17)**.** Use row operations to show that the determinants

$$\begin{vmatrix} 1 & a & b+c \\ 1 & b & a+c \\ 1 & c & a+b \end{vmatrix}$$

is zero.

Problem 5 (3.S.19)**.** Compute the determinant

$$\begin{vmatrix} 9 & 1 & 9 & 9 & 9 \\ 9 & 0 & 9 & 9 & 2 \\ 4 & 0 & 0 & 5 & 0 \\ 9 & 0 & 3 & 9 & 0 \\ 6 & 0 & 0 & 7 & 0 \end{vmatrix}$$

2. Additional Problems

Problem 6 (3.2.41)**.** Let U be a square matrix such that $U^T U = I$. Show that $\det U = \pm 1$.

Problem 7. Let $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$ be a 3×3 matrix where each $\mathbf{a}_i \in \mathbb{R}^3$. Can you find a matrix B such that $AB = \begin{bmatrix} \mathbf{a}_2 & \mathbf{a}_1 & \mathbf{a}_3 \end{bmatrix}$? What is the determinant of B?

Problem 1. If we change the linear system into a matrix multiplication, we have

$$\underbrace{\begin{bmatrix} 1 & 1 & 0 \\ -3 & 0 & 2 \\ 0 & 1 & -2 \end{bmatrix}}_{=A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}}_{=\mathbf{b}}$$

Then

$$A_1(\mathbf{b}) = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 0 & 2 \\ 2 & 1 & -2 \end{bmatrix}, \quad A_2(\mathbf{b}) = \begin{bmatrix} 1 & 3 & 0 \\ -3 & 0 & 2 \\ 0 & 2 & -2 \end{bmatrix}, \quad A_3(\mathbf{b}) = \begin{bmatrix} 1 & 1 & 3 \\ -3 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

We just now need to compute four determinants.

$$\det A_1(\mathbf{b}) = \begin{vmatrix} 3 & 1 & 0 \\ 0 & 0 & 2 \\ 2 & 1 & -2 \end{vmatrix} = -2 \begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} = -2$$

$$\det A_2(\mathbf{b}) = \begin{vmatrix} 1 & 3 & 0 \\ -3 & 0 & 2 \\ 0 & 2 & -2 \end{vmatrix} = -4 - 3 \cdot 6 = -22$$

$$\det A_3(\mathbf{b}) = \begin{vmatrix} 1 & 1 & 3 \\ -3 & 0 & 0 \\ 0 & 1 & 2 \end{vmatrix} = -(-3) \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} = -3$$

$$\det A = \begin{vmatrix} 1 & 1 & 0 \\ -3 & 0 & 2 \\ 0 & 1 & -2 \end{vmatrix} = -2 - 6 = -8$$

Therefore

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{11}{4} \\ \frac{3}{8} \end{bmatrix}$$

Problem 2. The volume is given by the absolute value of the determinant

$$\begin{vmatrix} 1 & 0 & -3 \\ 1 & 2 & 4 \\ 5 & 1 & 0 \end{vmatrix} = 1 \cdot (0 - 4) - 3(1 - 10) = -4 + 27 = 23$$

Problem 3. Let T be a linear transformation defined by $\mathbf{x} \mapsto A\mathbf{x}$.

$$(\text{area of } T(S)) = (\text{area of } S) \cdot |\det A|$$

The area of S is given by

$$\left| \det \begin{bmatrix} -2 & 3 \\ -2 & 5 \end{bmatrix} \right| = |-4| = 4$$

The determinant of A is

$$\begin{vmatrix} 6 & -3 \\ -3 & 2 \end{vmatrix} = 3$$

Therefore the area of $T(S)$ is $4 \cdot 3 = 12$.

Problem 4.

$$\begin{vmatrix} 1 & a & b+c \\ 1 & b & a+c \\ 1 & c & a+b \end{vmatrix} = \begin{vmatrix} 1 & a & b+c \\ 0 & b-a & a-b \\ 0 & c-a & a-c \end{vmatrix} = (b-a)(c-a)\begin{vmatrix} 1 & a & b+c \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{vmatrix}$$

$$= (b-a)(c-a)\begin{vmatrix} \mathbf{1} & a & b+c \\ 0 & \mathbf{1} & -1 \\ 0 & 0 & \mathbf{0} \end{vmatrix} = 0$$

Problem 5. We use column cofactor expansion

$$\begin{vmatrix} 9 & \mathbf{1} & 9 & 9 & 9 \\ 9 & 0 & 9 & 9 & 2 \\ 4 & 0 & 0 & 5 & 0 \\ 9 & 0 & 3 & 9 & 0 \\ 6 & 0 & 0 & 7 & 0 \end{vmatrix} = -\begin{vmatrix} 9 & 9 & 9 & \mathbf{2} \\ 4 & 0 & 5 & 0 \\ 9 & 3 & 9 & 0 \\ 6 & 0 & 7 & 0 \end{vmatrix} = 2\begin{vmatrix} 4 & 0 & 5 \\ 9 & \mathbf{3} & 9 \\ 6 & 0 & 7 \end{vmatrix} = 6\begin{vmatrix} 4 & 5 \\ 6 & 7 \end{vmatrix} = 6 \cdot (28-30) = -12$$

Problem 6.

$$\det(U^TU) = \det(I) \quad \Rightarrow \quad \det(U^T)\det(U) = 1 \quad \Rightarrow \quad \det(U)^2 = 1 \quad \Rightarrow \quad \det(U) = \pm 1$$

Problem 7. We write $B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix}$. Then

$$AB = A\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix} = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & A\mathbf{b}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{a}_2 & \mathbf{a}_1 & \mathbf{a}_3 \end{bmatrix}$$

Recall that if $\mathbf{e}_i$ is the standard (basis) vector where the ith entry is 1 and 0 everywhere else, $A\mathbf{e}_i = \mathbf{a}_i$. Therefore if $\mathbf{b}_1 = \mathbf{e}_2$, $\mathbf{b}_2 = \mathbf{e}_1$, and $\mathbf{b}_3 = \mathbf{e}_3$, then we obtain $AB = \begin{bmatrix} \mathbf{a}_2 & \mathbf{a}_1 & \mathbf{a}_3 \end{bmatrix}$. Therefore

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$