Worksheet 13	**Name:**	______________________
MATH 240 (Spring 2024)	**Section:**	0111(12PM-1PM) / 0121(1PM-2PM)
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1. Suggested Problems

In Problem 1-4, determine if the given set is a subspace of $\mathbb{P}_n$ for an appropriate value of n. Justify your answers.

Problem 1 (4.1.5)**.** All polynomials of the form $\boldsymbol{p}(t) = at^2$ where a is in $\mathbb{R}$.

Problem 2 (4.1.6)**.** All polynomials of the form $\boldsymbol{p}(t) = a + t^2$, where a is in $\mathbb{R}$.

Problem 3 (4.1.7)**.** All polynomials of degree at most 3, with integers as coefficients.

Problem 4 (4.1.8)**.** All polynomials in $\mathbb{P}_n$ such that $\boldsymbol{p}(0) = 0$.

Problem 5 (4.1.11)**.** Let W be the set of all vectors of the form

$$\begin{bmatrix} 5b + 2c \\ b \\ c \end{bmatrix}$$

where b and c are arbitrary. Find vectors $\mathbf{u}$ and $\mathbf{v}$ such that $W = \mathrm{Span}\{\mathbf{u}, \mathbf{v}\}$. Why does this show that W is a subspace of $\mathbb{R}^3$.

Problem 6 (4.1.16)**.** Let W be the set of all vectors of the form shown below, where a, b, and c represent arbitrary real numbers. In each case, either find a set S of vectors that span W or give an example to show that W is *not* a vector space.

$$\begin{bmatrix} -a + 1 \\ a - 6b \\ 2b + a \end{bmatrix}$$

Problem 7 (4.2.2)**.** Determine if $\mathbf{w} = \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix}$ is in $\mathrm{Nul}\ A$, where

$$A = \begin{bmatrix} 5 & 21 & 19 \\ 13 & 23 & 2 \\ 8 & 14 & 1 \end{bmatrix}.$$

In Problems 8-9, find an explicit description of $\mathrm{Nul}\ A$ by listing vectors that span the null space.

Problem 8 (4.2.3)**.**

$$A = \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{bmatrix}$$

Problem 9 (4.2.4)**.**

$$A = \begin{bmatrix} 1 & -6 & 4 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

For Problems 1-4, the given set will be called W.

Problem 1.

(a) $0 = 0 \cdot t^2 \in W$.

(b) Given arbitrary $a_1t^2, a_2t^2 \in W$, $a_1t^2 + a_2t^2 = (a_1 + a_2)t^2 \in W$.

(c) Given arbitrary $at^2 \in W$ and $c \in \mathbb{R}$, $c \cdot (at^2) = (ac)t^2 \in W$.

Therefore, W is a subspace of $\mathbb{P}_n$ for $n \geq 2$.

Problem 2. Since all elements in W has nonzero coeffient for the t^2-term, 0 is not in W. Therefore, W is not a subspace of any $\mathbb{P}_n$.

Problem 3. Consider $t \in W$. Even though $\sqrt{2} \in \mathbb{R}$, $\sqrt{2} \cdot t = \sqrt{2}t$ is not in W. Therefore, W is not closed under scalar multiplication. W is not a subspace of any $\mathbb{P}_n$.

Problem 4.

(a) Let 0 be the zero polynomial. Then by definition $0(0) = 0$, so $0 \in W$.

(b) Let $p_1, p_2 \in W$. Then

$$(p_1 + p_2)(0) = p_1(0) + p_2(0) = 0 + 0 = 0$$

Therefore, $p_1 + p_2 \in W$.

(c) Let $p \in W$ and $c \in \mathbb{R}$. Then $(cp)(0) = c \cdot p(0) = c \cdot 0 = 0$. In particular, $cp \in W$.

This shows that W is a subspace of $\mathbb{P}_n$.

Problem 5. Any elements in W is of the form

$$\begin{bmatrix} 5b + 2c \\ b \\ c \end{bmatrix} = b \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

Therefore, $W = \operatorname{Span}\left\{ \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$. By Theorem 1 of Chapter 4, W is a subspace of $\mathbb{R}^3$.

Problem 6. Suppose zero vector is in W. Then there exists $a, b \in \mathbb{R}$ such that

$$\begin{bmatrix} -a + 1 \\ a - 6b \\ 2b + a \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By the first entry, we get $a = 1$. Then by second entry $b = \frac{1}{6}$. However,

$$2b + a = \frac{1}{3} + 1 = \frac{4}{3} \neq 0$$

Therefore, the zero vector is not in W. This shows that W is not a subspace of $\mathbb{R}^3$.

Problem 7. To check $\mathbf{w} \in \operatorname{Nul} A$. We simply need to check that $A\mathbf{w} = 0$ which is indeed true.

$$A\mathbf{w} = \begin{bmatrix} 5 & 21 & 19 \\ 13 & 23 & 2 \\ 8 & 14 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 25 - 63 + 38 \\ 65 - 69 + 4 \\ 40 - 42 + 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For problem 8-9, we first find general solutions to the associated homogeneous linear systems.

Problem 8. Since A is already echelon form, we know that x_3 and x_4 are free variables, and we can write the general solutions as

$$\begin{bmatrix} -3x_2 - 5x_3 \\ -4x_3 + 2x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 12x_3 - 6x_4 - 5x_3 \\ -4x_3 + 2x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7x_3 - 6x_4 \\ -4x_3 + 2x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 7 \\ -4 \\ 0 \\ 1 \end{bmatrix} + x_4 \begin{bmatrix} -6 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

In particular,

$$\text{Nul } A = \text{Span} \left\{ \begin{bmatrix} 7 \\ -4 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ 2 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Problem 9. By the second row, we have $x_3 = 0$. Then x_2 and x_4 are free variables with

$$\begin{bmatrix} 6x_2 - 4x_3 \\ x_2 \\ 0 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6x_2 \\ x_2 \\ 0 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 6 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Therefore,

$$\text{Nul } A = \text{Span} \left\{ \begin{bmatrix} 6 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$