Worksheet 14	**Name:**	______________________
MATH 240 (Spring 2024)	**Section:**	0111(12PM-1PM) / 0121(1PM-2PM)
Mar 14th, 2024	**TA:**	Shin Eui Song

1. Suggested Problems

Problem 1 (4.3.1, 4.3.3, 4.3.5, 4.3.6)**.** Determine which sets are bases for $\mathbb{R}^3$. Of the sets that are not bases, determine which ones are linearly independendent and which ones span $\mathbb{R}^3$. Justify your answers.

a)

$$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}$$

b)

$$\begin{bmatrix}1\\0\\-2\end{bmatrix}, \begin{bmatrix}3\\2\\-4\end{bmatrix}, \begin{bmatrix}-3\\-5\\1\end{bmatrix}$$

c)

$$\begin{bmatrix}1\\-3\\0\end{bmatrix}, \begin{bmatrix}-2\\9\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\-3\\5\end{bmatrix}$$

d)

$$\begin{bmatrix}1\\2\\-3\end{bmatrix}, \begin{bmatrix}-4\\-5\\6\end{bmatrix}$$

Problem 2 (4.3.16)**.** Find a basis for the space spanned by the given vectors, $v_1, \ldots, v_5$.

$$\begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}-2\\1\\-1\\1\end{bmatrix}, \begin{bmatrix}6\\-1\\2\\-1\end{bmatrix}, \begin{bmatrix}5\\-3\\3\\-4\end{bmatrix}, \begin{bmatrix}0\\3\\-1\\1\end{bmatrix}$$

Problem 3 (4.3.26)**. (T/F)** A basis is a linearly independent set that is as large as possible.

Problem 4 (4.3.30)**. (T/F)** If B is an echelon form of a matrix A, then the pivot columns of B form a basis for $\operatorname{Col} A$.

Problem 5 (4.3.32)**. (T/F)** If A and B are row equivalent, then their row spaces are the same.

2. Additional Problems

Problem 6 (4.4.3)**.** Find the vector x determined by the given coordinate vector $[x]_{\mathscr{B}}$ and the given basis $\mathscr{B}$.

$$\mathscr{B} = \left\{ \begin{bmatrix}1\\-4\\3\end{bmatrix}, \begin{bmatrix}5\\2\\-2\end{bmatrix}, \begin{bmatrix}4\\-7\\0\end{bmatrix} \right\}, \quad [x]_{\mathscr{B}} = \begin{bmatrix}3\\0\\-1\end{bmatrix}$$

Problem 7 (4.4.5)**.** Find the coordinate vector $[x]_{\mathscr{B}}$ of x relative to the basis $\mathscr{B}$ where

$$b_1 = \begin{bmatrix}1\\-3\end{bmatrix}, b_2 = \begin{bmatrix}2\\-5\end{bmatrix}, x = \begin{bmatrix}-2\\1\end{bmatrix}$$

Problem 1.

a) The matrix whose column vectors are given in the problem is

$$\begin{bmatrix} \mathbf{1} & 1 & 1 \\ 0 & \mathbf{1} & 1 \\ 0 & 0 & \mathbf{1} \end{bmatrix}$$

This matrix has pivots in every row and every column. Therefore, this set is a basis for $\mathbb{R}^3$.

b) The matrix whose column vectors are given in the problem is

$$\begin{bmatrix} 1 & 3 & -3 \\ 0 & 2 & -5 \\ -2 & -4 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -3 \\ 0 & 2 & -5 \\ 0 & 2 & -5 \end{bmatrix} \to \begin{bmatrix} \mathbf{1} & 3 & -3 \\ 0 & \mathbf{2} & -5 \\ 0 & 0 & 0 \end{bmatrix}$$

The matrix does not have pivots every row, so the set is not a basis.

c) The set contains four vectors in $\mathbb{R}^3$. This means that the set is linearly dependent, and this means that the set is not a basis.

d) The matrix whose column vectors are given in the problem is

$$\begin{bmatrix} 1 & -4 \\ 2 & -5 \\ -3 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & -4 \\ 0 & 3 \\ 0 & -6 \end{bmatrix} \to \begin{bmatrix} \mathbf{1} & -4 \\ 0 & \mathbf{3} \\ 0 & 0 \end{bmatrix}$$

Since every row does not have a pivot, the set is not a basis.

Problem 2. We reduce the following matrix to echelon form to find the pivot columns.

$$\begin{bmatrix} 1 & -2 & 6 & 5 & 0 \\ 0 & 1 & -1 & -3 & 3 \\ 0 & -1 & 2 & 3 & -1 \\ 1 & 1 & -1 & -4 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 6 & 5 & 0 \\ 0 & 1 & -1 & -3 & 3 \\ 0 & -1 & 2 & 3 & -1 \\ 0 & 3 & -7 & -9 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 6 & 5 & 0 \\ 0 & 1 & -1 & -3 & 3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & -4 & 0 & -8 \end{bmatrix}$$
$$\to \begin{bmatrix} \mathbf{1} & -2 & 6 & 5 & 0 \\ 0 & \mathbf{1} & -1 & -3 & 3 \\ 0 & 0 & \mathbf{1} & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The pivot columns are column 1, 2, and 3. Therefore, $\{v_1, v_2, v_3\}$ form a basis for the space.

Problem 3. This is **True**. Read page 228 of the textbook.

Problem 4. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \to B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Then B is an echelon form of A and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is the pivot column of B. However,

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \notin \operatorname{Col} A = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} a \\ a \end{bmatrix} \,\middle|\, a \in \mathbb{R} \right\}$$

So this is **False**.

Problem 5. This is **True**. See Theorem 7 of Chapter 4.

Problem 6.

$$[x]_{\mathscr{B}} = 3 \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix} + 0 \cdot \begin{bmatrix} 5 \\ 2 \\ -2 \end{bmatrix} - \begin{bmatrix} 4 \\ -7 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -5 \\ 9 \end{bmatrix}$$

Problem 7. We need to solve a linear system whose augmented matrix is

$$\begin{bmatrix} 1 & 2 & -2 \\ -3 & -5 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & -5 \end{bmatrix}$$

Therefore $x_2 = -5$ and $x_1 = 8$. We thus have

$$[x]_{\mathscr{B}} = \begin{bmatrix} 8 \\ -5 \end{bmatrix}$$