Worksheet 15	**Name:**	______________________________
MATH 240 (Spring 2024)	**Section:**	0111(12PM-1PM) / 0121(1PM-2PM)
Mar 26th, 2024	**TA:**	Shin Eui Song

1. **Suggested Problems**

Problem 1 (4.4.7)**.** Find the coordinate vector $[x]_{\mathscr{B}}$ of x relative to the given basis $\mathscr{B} = \{b_1, \ldots, b_n\}$ where

$$b_1 = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}, \quad b_2 = \begin{bmatrix} -3 \\ 4 \\ 9 \end{bmatrix}, \quad b_3 = \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}, \quad x = \begin{bmatrix} 8 \\ -9 \\ 6 \end{bmatrix}$$

Problem 2 (4.4.31)**.** Use coordinate vectors to test the linear independence of the sets of polynomials. Explain your work.

$$\{1 + 2t^3, 2 + t - 3t^2, -t + 2t^2 - t^3\}$$

Problem 3. Determine whether the sets of polynomials form a basis for $\mathbb{P}_3$. Justify your conclusions.

$$\{3 + 7t, 5 + t - 2t^3, t - 2t^2, 1 + 16t - 6t^2 + 2t^3\}$$

Problem 4 (4.5.6)**.** For the following subspace, (a) find a basis, and (b) state the dimension.

$$\left\{ \begin{bmatrix} 3a + 6b - c \\ 6a - 2b - 2c \\ -9a + 5b + 3c \\ -3a + b + c \end{bmatrix} : a, b, c \text{ in } \mathbb{R} \right\}$$

Problem 5 (4.5.9)**.** Find the dimension of the subspace spanned by the given vectors

$$\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} -7 \\ -3 \\ 1 \end{bmatrix}$$

Problem 6 (4.5.19)**. (T/F)** A plane in $\mathbb{R}^3$ is a two-dimensional subspace of $\mathbb{R}^3$.

Problem 7 (4.5.22)**. (T/F)** The dimensions of the row space and the column space of A are the same, even if A is not square.

Problem 8 (4.5.23)**. (T/F)** If B is any echelon form of A, then the pivot columns of B form a basis for the column space of A.

Problem 1. We need to solve a linear system whose augmented matrix is

$$\begin{bmatrix} 1 & -3 & 2 & 8 \\ -1 & 4 & -2 & -9 \\ -3 & 9 & 4 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 2 & 8 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 10 & 30 \end{bmatrix}$$

Then $10x_3 = 30 \Rightarrow x_3 = 10$, $x_2 = -1$. Since $x_1 - 3x_2 + 2x_3 = 8$, we have $x_1 = -1$. Therefore

$$[x]_{\mathscr{B}} = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$$

Problem 2. We use the basis $\{1, t, t^2, t^3\}$. Then the coordinate vector of $1 + 2t^3$ is $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}$. Then the matrix whose columns are the coordinate vectors of the given set is

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & -3 & 2 \\ 2 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & -3 & 2 \\ 0 & -4 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & -5 \end{bmatrix} \to \begin{bmatrix} \mathbf{1} & 2 & 0 \\ 0 & \mathbf{1} & -1 \\ 0 & 0 & \mathbf{-1} \\ 0 & 0 & 0 \end{bmatrix}$$

Since there are pivots in every column, the vectors are linearly independent.

Problem 3. The matrix whose columns are the coordinate vectors of the given set is

$$\begin{bmatrix} 3 & 5 & 0 & 1 \\ 7 & 1 & 1 & 16 \\ 0 & 0 & -2 & -6 \\ 0 & -2 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 3 & 5 & 0 & 1 \\ 1 & -9 & 1 & 14 \\ 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -9 & 1 & 14 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \\ 3 & 5 & 0 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -9 & 1 & 14 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 32 & -3 & -41 \end{bmatrix} \to \begin{bmatrix} 1 & -9 & 1 & 14 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & -3 & -9 \end{bmatrix} \to \begin{bmatrix} \mathbf{1} & -9 & 1 & 14 \\ 0 & \mathbf{1} & 0 & -1 \\ 0 & 0 & \mathbf{1} & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The coordinate vectors are linearly dependent as not every column has a pivot. The given set is not a basis for $\mathbb{P}_3$.

Problem 4. The given set can be rewritten as

$$\text{Span}\left\{ \begin{bmatrix} 3 \\ 6 \\ -9 \\ -3 \end{bmatrix}, \begin{bmatrix} 6 \\ -2 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 3 \\ 1 \end{bmatrix} \right\} = \text{col} \begin{bmatrix} 3 & 6 & -1 \\ 6 & -2 & -2 \\ -9 & 5 & 3 \\ -3 & 1 & 1 \end{bmatrix}$$

A basis of a column space is formed by pivot columns of the matrix. As

$$\begin{bmatrix} 3 & 6 & -1 \\ 6 & -2 & -2 \\ -9 & 5 & 3 \\ -3 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} \mathbf{3} & 6 & -1 \\ 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore, the first two columns are the pivot columns. A basis of the subspace is

$$\left\{ \begin{bmatrix} 3 \\ 6 \\ -9 \\ -3 \end{bmatrix}, \begin{bmatrix} 6 \\ -2 \\ 5 \\ 1 \end{bmatrix} \right\}$$

Since there are two vectors in a basis, the dimension is 2.

Problem 5. Similar to Problem 4, we find the pivot positions of the matrix

$$\begin{bmatrix} 1 & 3 & 9 & -7 \\ 0 & 1 & 4 & -3 \\ 2 & 1 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} \mathbf{1} & 3 & 9 & -7 \\ 0 & \mathbf{1} & 4 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there are two vectors in a basis, the dimension is 2.

Problem 6. This is **True** by definition.

Problem 7. This is **True** because

$$\dim \text{Col } A = \# \text{ of pivot columns } = \# \text{ of pivot rows } = \dim \text{Row } A$$

Problem 8. This is **False**. This also appeared in Chapter 4, Section 3, Problem 30. Consider

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \to B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

The pivot column $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ of B cannot span

$$\text{Col } A = \left\{ \begin{bmatrix} a \\ a \end{bmatrix} \middle| \, a \in \mathbb{R} \right\}$$

because $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is not in $\text{Col } A$.