Worksheet 16 | **Name:** ______________________
MATH 240 (Spring 2024) | **Section:** 0111(12PM-1PM) / 0121(1PM-2PM)
Mar 28th, 2024 | **TA:** Shin Eui Song

1. **Suggested Problems**

Problem 1 (4.5.34). If a 6×3 A has rank 3, find nullity A, rank A, and rank A^T.

Problem 2 (4.5.35). Suppose a 4×7 matrix A has four pivot columns. Is $\mathrm{Col}\, A = \mathbb{R}^4$? Is $\mathrm{Nul}\, A = \mathbb{R}^3$? Explain your answers.

Problem 3 (4.5.36). Suppose a 5×6 matrix A has four pivot columns. What is nullity A? Is $\mathrm{Col}\, A = \mathbb{R}^4$? Why or why not?

Problem 4 (4.6.1). Let $\mathscr{B} = \{b_1, b_2\}$ and $\mathscr{C} = \{c_1, c_2\}$ be bases for a vector space V, and suppose $b_1 = 6c_1 - 2c_2$ and $b_2 = 9c_1 - 4c_2$.

(a) Find the change-of-coordinates matrix from $\mathscr{B}$ to $\mathscr{C}$.

(b) Find $[x]_{\mathscr{C}}$ for $x = -3b_1 + 2b_2$. Use part (a).

Problem 5 (4.6.3). Let $\mathscr{U} = \{u_1, u_2\}$ and $\mathscr{W} = \{w_1, w_2\}$ be bases for V, and let P be a matrix whose columns are $[u_1]_{\mathscr{W}}$ and $[u_2]_{\mathscr{W}}$. Which of the following equations is satisfied by P for all x in V?

a) $[x]_{\mathscr{U}} = P[x]_{\mathscr{W}}$ b) $[x]_{\mathscr{W}} = P[x]_{\mathscr{U}}$

Problem 6 (4.6.7). Let $\mathscr{B} = \{b_1, b_2\}$ and $\mathscr{C} = \{c_1, c_2\}$ be bases for $\mathbb{R}^2$. Find the change-of-coordinates matrix from $\mathscr{B}$ to $\mathscr{C}$ and the change-of-coordinates matrix from $\mathscr{C}$ to $\mathscr{B}$.

$$b_1 = \begin{bmatrix} 7 \\ 5 \end{bmatrix}, \quad b_2 = \begin{bmatrix} -3 \\ -1 \end{bmatrix}, \quad c_1 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}, \quad c_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

Problem 1. By rank theorem, $\text{nullity } A = (\text{number of columns}) - \text{rank } A = 3 - 3 = 0$. Also, $\text{rank } A = \text{rank } A^T = 3$.

Problem 2. As the number of pivot columns is the rank of A, it follows that $\text{Col } A$ is a 4-dimensional subspace of $\mathbb{R}^4$. Therefore $\text{Col } A = \mathbb{R}^4$. On the other hand, we only know that $\text{Null } A$ is a 3-dimensional subspace of $\mathbb{R}^7$ where $3 = 7 - \text{Rank } A$. Therefore, $\text{Null } A \neq \mathbb{R}^3$.

Problem 3. By rank theorem, $\text{Nullity } A = 6 - 4 = 2$. Though $\dim \text{Col } A = 4$, it is a subspace of $\mathbb{R}^5$, so $\text{Col } A \neq \mathbb{R}^4$.

Problem 4.

(a) Recall that the change-of-coordinate matrix is given by

$$P_{\mathscr{C}\leftarrow\mathscr{B}} = \begin{bmatrix} [b_1]_{\mathscr{C}} & [b_2]_{\mathscr{C}} \end{bmatrix} = \begin{bmatrix} 6 & 9 \\ -2 & -4 \end{bmatrix}$$

(b)

$$[x]_{\mathscr{C}} = P_{\mathscr{C}\leftarrow\mathscr{B}}[x]_{\mathscr{B}} = \begin{bmatrix} 6 & 9 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

Problem 5. Since $P = \begin{bmatrix} [u_1]_{\mathscr{W}} & [u_2]_{\mathscr{W}} \end{bmatrix}$, we have

$$P \begin{bmatrix} 1 \\ 0 \end{bmatrix} = [u_1]_{\mathscr{W}} \text{ and } P \begin{bmatrix} 0 \\ 1 \end{bmatrix} = [u_2]_{\mathscr{W}}$$

Then it follows that $P = P_{\mathscr{W}\leftarrow\mathscr{U}}$, with

$$[x]_{\mathscr{W}} = P[x]_{\mathscr{U}}$$

Therefore equation b is satisfied.

Problem 6. We have

$$\begin{bmatrix} c_1 & c_2 \mid b_1 & b_2 \end{bmatrix} \to \begin{bmatrix} I \mid P_{\mathscr{C}\leftarrow\mathscr{B}} \end{bmatrix}$$

Then as

$$\begin{bmatrix} 1 & -2 & 7 & -3 \\ -5 & 2 & 5 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 7 & -3 \\ 0 & -8 & 40 & -16 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -2 & 7 & -3 \\ 0 & 1 & -5 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 & 1 \\ 0 & 1 & -5 & 2 \end{bmatrix}$$

We have

$$P_{\mathscr{C}\leftarrow\mathscr{B}} = \begin{bmatrix} -3 & 1 \\ -5 & 2 \end{bmatrix}$$

Similarly, we have

$$\begin{bmatrix} 7 & -3 & 1 & -2 \\ 5 & -1 & -5 & 2 \end{bmatrix} \to \begin{bmatrix} 2 & -2 & 6 & -4 \\ 5 & -1 & -5 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 3 & -2 \\ 5 & -1 & -5 & -2 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -1 & 3 & -2 \\ 0 & 4 & -20 & 12 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 3 & -2 \\ 0 & 1 & -5 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & -5 & 3 \end{bmatrix}$$

We conclude that

$$P_{\mathscr{B}\leftarrow\mathscr{C}} = \begin{bmatrix} -2 & 1 \\ -5 & 3 \end{bmatrix}$$