Worksheet 18		**Name:**	______________________
MATH 240 (Spring 2024)		**Section:**	0111(12PM-1PM) / 0121(1PM-2PM)
Apr 8th, 2024		**TA:**	Shin Eui Song

1. **Suggested Problems**

Problem 1 (5.2.6)**.** Find the characteristic polynomial and the eigenvalues of the matrix

$$\begin{bmatrix} 1 & -4 \\ 4 & 6 \end{bmatrix}$$

Problem 2 (5.2.16)**.** List the eigenvalues, repeated according to their multiplicities

$$\begin{bmatrix} 4 & -7 & 0 & 2 \\ 0 & 3 & -4 & 6 \\ 0 & 0 & 3 & -8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Problem 3 (5.2.19)**.** Let A be an $n \times n$ matrix, and suppose A has n real eigenvalues, $\lambda_1, \dots, \lambda_n$, repeated according to multiplicities, so that $\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda)$. Explain why $\det A$ is the product of the n eigenvalues of A. (This result is true for any square matrix when complex eigenvalues are considered.)

Problem 4 (5.2.24)**.** **(T/F)** The matrices A and $B^{-1}AB$ have the same set of eigenvalues for every invertible matrix B.

Problem 5 (5.2.25)**.** **(T/F)** If 2 is an eigenvalue of A, then $A - 2I$ is not invertible.

Problem 6 (5.2.26)**.** **(T/F)** If two matrices have the same set of eigenvalues, then they are similar.

Problem 1. The characteristic polynomial is

$$\det \begin{bmatrix} 1-\lambda & -4 \\ 4 & 6-\lambda \end{bmatrix} = (1-\lambda)(6-\lambda) - (-4)\cdot 4 = \lambda^2 - 7\lambda + 22$$

The *discriminant* $(-7)^2 - 4 \cdot 1 \cdot 22 < 0$, so there are no real eigenvalues. The complex eigenvalues (if needed) are given by the quadratic formula

$$\lambda = \frac{7 \pm \sqrt{49-88}}{2} = \frac{7 \pm \sqrt{-39}}{2}.$$

Problem 2. The characteristic polynomial is

$$\det \begin{bmatrix} 4-\lambda & -7 & 0 & 2 \\ 0 & 3-\lambda & -4 & 6 \\ 0 & 0 & 3-\lambda & -8 \\ 0 & 0 & 0 & 1-\lambda \end{bmatrix} = (4-\lambda)(3-\lambda)(3-\lambda)(1-\lambda)$$

Here we used the fact that the determinant of a triangular matrix is the product of the entries along the diagonal. Then the eigenvalue of this matrix is given by $4, 3, 3, 1$ according to their multiplicities.

Problem 3. Let $\lambda = 0$. Then the left-hand side becomes

$$\det(A - \lambda I) = \det(A)$$

and the right-hand side becomes

$$(\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda) = \lambda_1 \cdots \lambda_2 \cdots \lambda_n$$

Therefore $\det(A)$ is the product of the n eigenvalues of A.

Problem 4. (True) By definition A and $B^{-1}AB$ are similar for every invertible matrix B. By Theorem 4 (pg 286), they have the same sets of eigenvalues.

Problem 5. (True) If 2 is an eigenvalue of A, then $\mathrm{Null}(A - 2I)$ is not the zero space. Therefore, $A - 2I$ is not invertible.

Problem 6. (False) Consider

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Both have characteristic polynomial $(1-\lambda)^2$, so the eigenvalue is 1 with multiplicity 2 for both. However, for every invertible matrix P,

$$P^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so the matrices cannot be similar.