Worksheet 19	**Name:**	______________________
MATH 240 (Spring 2024)	**Section:**	0111(12PM-1PM) / 0121(1PM-2PM)
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1. **Suggested Problems**

Problem 1 (5.3.1)**.** Let $A = PDP^{-1}$. Compute A^4 where

$$P = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix} \text{ and } D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

Problem 2 (5.3.5)**.** The matrix A is factored in the form PDP^{-1}. Use the Diagonalization Theorem to find the eigenvalues of A and a basis for each eigenspace.

$$\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/4 & 1/2 & -3/4 \\ 1/4 & -1/2 & 1/4 \end{bmatrix}$$

Problem 3 (5.3.7)**.** Diagonalize the matrix if possible.

$$\begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$$

For Problems 4-6, A, P, and D are $n \times n$ matrices. Explain why the statment is True or False.

Problem 4 (5.3.22)**.** **(T/F)** If $\mathbb{R}^n$ has a basis of eigenvectors of A, then A is diagonalizable.

Problem 5 (5.3.23)**.** **(T/F)** A is diagonalizable if and only if A has n eigenvalues, counting multiplicities.

Problem 6 (5.3.24)**.** **(T/F)** If A is diagonalizable, then A is invertible.

Problem 1. If $A = PDP^{-1}$, then $A^4 = PD^4P^{-1}$. For the power of diagonal matrices, one can simply raise the power of the diagonal entries, i.e.

$$D^4 = \begin{bmatrix} 2^4 & 0 \\ 0 & 1^4 \end{bmatrix} = \begin{bmatrix} 16 & 0 \\ 0 & 1 \end{bmatrix}$$

As $\det(P) = 1$, $P^{-1} = \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}$. Therefore

$$A^4 = PD^4P^{-1} = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 16 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 226 & -525 \\ 90 & -209 \end{bmatrix}$$

Problem 2. The eigenvalues are given by the digonal entries of D, i.e.,

$$5, 1, 1$$

are the eigenvalues of A with multiplicity. Then the columns of P gives the corresponding eigenvectors that form the basis of the corresponding eigenspace, i.e.

$$\underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{\text{for } \lambda=5}, \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}}_{\text{for } \lambda=1}$$

Problem 3.

(a) We first find the eigenvalues. The characteristic polynomial is

$$(1-\lambda)(-1-\lambda) = -(1-\lambda)(1+\lambda)$$

Therefore, the eigenvalues are ± 1. Since there are 2 distinct eigenvalues for 2×2 matrix, the matrix is diagonalizable.

(b) We need to find basis for the eigenspace of 1 and the eigenspace of -1.

$$A - I_2 = \begin{bmatrix} 0 & 0 \\ 6 & -2 \end{bmatrix}$$

so

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

works.

$$A + I_2 = \begin{bmatrix} 2 & 0 \\ 6 & 0 \end{bmatrix}$$

so

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

works.

(c) The digaonalization is

$$\begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

Problem 4. **True**. This is the diagonalization theorem (pg 291).

Problem 5. **False**. Consider

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

The characteristic polynomial is $(1-\lambda)^2$ and the eigenspace for 1 is given by

$$E_1 = \text{Null}(A - \lambda I_2) = \text{Null}\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \; y = 0 \right\}$$

The general solution is given by

$$\begin{bmatrix} x_1 \\ 0 \end{bmatrix} \rightsquigarrow E_1 = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

which is of dimension 1. Therefore, by Theorem 7 (pg. 294), the matrix is not diagonalizable.

Problem 6. False. Consider the diagonal matrix with 0 on one of the diagonal entry, e.g.

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

The matrix is a diagonal matrix, so it is diagonalizable ($P = I_2$) and is not invertible.