Worksheet 2 **Name:** ______________________
MATH 240 (Spring 2024) **Section:** 0111(12PM-1PM) / 0121(1PM-2PM)
Feb 2, 2024 **TA:** Shin Eui Song

1. Suggested Problems

Problem 1 (1.2.1)**.** Determine which matrices are in reduced echelon form and which others are only in echlon form.

a)
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

b)
$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

c)
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

d)
$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

Problem 2 (1.2.9)**.** Find the general solution of the system whose augmented matrix is
$$\begin{bmatrix} 0 & 1 & -6 & 5 \\ 1 & -2 & 7 & -4 \end{bmatrix}$$

Problem 3 (1.2.11)**.** Find the general solution of the system whose augmented matrix is
$$\begin{bmatrix} 3 & -4 & 2 & 0 \\ -9 & 12 & -6 & 0 \\ -6 & 8 & -4 & 0 \end{bmatrix}$$

Problem 4 (1.2.22)**.** Determine the value(s) of h such that the matrix
$$\begin{bmatrix} 1 & -3 & -2 \\ 5 & h & -7 \end{bmatrix}$$
is the augmented matrix of a consistent linear system.

2. Additional Problems

Problem 5. Consider the linear system whose augmented matrix is
$$\begin{bmatrix} 1 & 0 & -2 & 1 & 4 \\ 1 & 1 & -3 & 0 & 6 \\ 2 & 1 & -5 & 1 & 10 \\ 3 & 0 & -6 & 3 & 12 \end{bmatrix}$$
Find the solution of the linear system where $x_1 = 0$ and $x_2 = 3$.

Worksheet 1 Solution
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Problem 1.

a) reduced echelon

b) echelon

c) reduced echelon

d) echelon

Problem 2.

$$\begin{bmatrix} 0 & 1 & -6 & 5 \\ 1 & -2 & 7 & -4 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & -2 & 7 & -4 \\ 0 & 1 & -6 & 5 \end{bmatrix} \xrightarrow{R_1 = R_1 + 2R_2} \begin{bmatrix} \color{blue}{1} & 0 & -5 & 6 \\ 0 & \color{blue}{1} & -6 & 5 \end{bmatrix}$$

The blue entries of the last matrix indicates pivots of the augmented matrix. Therefore x_1 and x_2 are leading variables, and x_3 and x_4 are free variables. We obtain a system

$$\begin{aligned} x_1 \qquad - 5x_3 &= 6 \\ x_2 - 6x_3 &= 5 \end{aligned}$$

Hence, the general solution is

$$\begin{aligned} x_1 &= 6 + 5x_3 \\ x_2 &= 5 + 6x_3 \\ x_3 &\text{ is free} \end{aligned}$$

Problem 3.

$$\begin{bmatrix} 3 & -4 & 2 & 0 \\ -9 & 12 & -6 & 0 \\ -6 & 8 & -4 & 0 \end{bmatrix} \xrightarrow[R_3 = R_3 + 2R_1]{R_2 = R_2 + 3R_1} \begin{bmatrix} \color{blue}{3} & -4 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore, the equation we have is $3x_1 - 4x_2 + 2x_3 = 0$, or $3x_1 = 4x_2 - 2x_3$. Therefore, the general solution is

$$\begin{aligned} x_1 &= \tfrac{4}{3}x_2 - \tfrac{2}{3}x_3 \\ x_2 &\text{ is free} \\ x_3 &\text{ is free} \end{aligned}$$

Problem 4.

$$\begin{bmatrix} 1 & -3 & -2 \\ 5 & h & -7 \end{bmatrix} \xrightarrow{R_2 = R_2 - 5R_1} \begin{bmatrix} 1 & -3 & -2 \\ 0 & h + 15 & 3 \end{bmatrix}$$

We use the theorem that says the following

Theorem. *A linear system is inconsistent if and only if the last column of the augmented matrix is a pivot column.*

Then the last column is a pivot column if and only if $h + 15 = 0$ if and only if $h = -15$. Therefore, the system is consistent if and only if $h \neq -15$.

Problem 5.

Note. This is not a good problem. The problem should have been simply find the general solution of the system whose augmented matrix is given. The first two row gives the equation $x_1 - 2x_3 + x_4 = 4$ and $x_1 + x_2 - 3x_3 = 6$. By plugging in x_1 and x_2, we get a system of linear equations that is easy to solve.

Here, we only explain how to find the general solution.

$$\begin{bmatrix} 1 & 0 & -2 & 1 & 4 \\ 1 & 1 & -3 & 0 & 6 \\ 2 & 1 & -5 & 1 & 10 \\ 3 & 0 & -6 & 3 & 12 \end{bmatrix} \xrightarrow[\substack{R_3 = R_3 - 2R_1 \\ R_4 = R_4 - 3R_1}]{R_2 = R_2 - R_1} \begin{bmatrix} 1 & 0 & -2 & 1 & 4 \\ 0 & 2 & -2 & -2 & 4 \\ 0 & 1 & -1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_3 = R_3 - \frac{1}{2}R_2]{R_2 = \frac{1}{2}R_2} \begin{bmatrix} 1 & 0 & -2 & 1 & 4 \\ 0 & 1 & -1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then we have

$$\begin{aligned} x_1 \quad\quad - 2x_3 + x_4 &= 4 \\ x_2 - \quad x_3 - x_4 &= 2 \end{aligned}$$

Therefore, the general solution is

$$\begin{array}{rcl} x_1 & = & 4 + 2x_3 - x_4 \\ x_2 & = & 2 + x_3 + x_4 \\ x_3 & \text{is free} & \\ x_4 & \text{is free} & \end{array}$$