Worksheet 20	**Name:**	______________________
MATH 240 (Spring 2024)	**Section:**	0111(12PM-1PM) / 0121(1PM-2PM)
Apr 16th, 2024	**TA:**	Shin Eui Song

1. **Suggested Problems**

Problem 1 (5.4.3)**.** Assume the mapping $T : \mathbb{P}_2 \to \mathbb{P}_2$ defined by

$$T(a_0 + a_1 t + a_2 t^2) = 3a_0 + (5a_0 - 2a_1)t + (4a_1 + a_2)t^2$$

is linear. Find the matrix representation of T relative to the basis $\mathscr{B} = \{1, t, t^2\}$.

Problem 2 (5.4.5)**.** Let $\mathscr{B} = \{b_1, b_2, b_3\}$ be a basis for a vector space V. Find $T(3b_1 - 4b_2)$ when T is a linear transformation from V to V whose matrix relative to $\mathscr{B}$ is

$$[T]_{\mathscr{B}} = \begin{bmatrix} 0 & -6 & 1 \\ 0 & 5 & -1 \\ 1 & -2 & 7 \end{bmatrix}$$

Problem 3 (5.4.10)**.** Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x) = Ax$ with

$$A = \begin{bmatrix} 5 & -3 \\ -7 & 1 \end{bmatrix}$$

Find a basis $\mathscr{B}$ for $\mathbb{R}^2$ with the property that $[T]_{\mathscr{B}}$ is diagonal.

Provide justification for each statement in Problems 4-6. In each case, the matrices are square.

Problem 4 (5.4.22)**.** If A is similar to B, then A^2 is similar to B^2.

Problem 5 (5.4.23)**.** If B is similar to A and C is similar to A, then B is similar to C.

Problem 6 (5.4.24)**.** If A is diagonalizable and B is similar to A, then B is also diagonalizable.

Problem 1. The matrix representation $[T]_{\mathscr{B}}$ of T relative to $\mathscr{B}$ is given by

$$\begin{bmatrix}[T(1)]_{\mathscr{B}} & [T(t)]_{\mathscr{B}} & [T(t^2)]_{\mathscr{B}}\end{bmatrix}$$

$$\begin{aligned} T(1) &= 3+5t &\Rightarrow [T(1)])_{\mathscr{B}} &= \begin{bmatrix}3\\5\\0\end{bmatrix} \\ T(t) &= -2t+4t^2 &\Rightarrow [T(t)]_{\mathscr{B}} &= \begin{bmatrix}0\\-2\\4\end{bmatrix} \\ T(t^2) &= t^2 &\Rightarrow [T(t^2)])_{\mathscr{B}} &= \begin{bmatrix}0\\0\\1\end{bmatrix} \end{aligned}$$

Therefore,

$$[T]_{\mathscr{B}} = \begin{bmatrix}3 & 0 & 0\\ 5 & -2 & 0\\ 0 & 4 & 1\end{bmatrix}$$

Problem 2. The $\mathscr{B}$-coordinate of $T(3b_1 - 4b_2)$ is given by

$$[T(3b_1-4b_2)]_{\mathscr{B}} = [T]_{\mathscr{B}}[3b_1-4b_2]_{\mathscr{B}} = \begin{bmatrix}0 & -6 & 1\\ 0 & 5 & -1\\ 1 & -2 & 7\end{bmatrix}\begin{bmatrix}3\\-4\\0\end{bmatrix} = \begin{bmatrix}24\\-20\\11\end{bmatrix}$$

Therefore, $T(3b_1-4b_2) = 24b_1 - 20b_2 + 11b_3$.

Problem 3. We diagonalize A. The characteristic polynomial is given by

$$\det\begin{bmatrix}5-\lambda & -3\\ -7 & 1-\lambda\end{bmatrix} = (5-\lambda)(1-\lambda) - 21 = \lambda^2 - 6\lambda - 16 = (\lambda-8)(\lambda+2)$$

Hence the eigenvalues are $\lambda = 8, -2$.

$$\begin{aligned} A - 8I &= \begin{bmatrix}-3 & -3\\ -7 & -7\end{bmatrix} \to \begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix} \rightsquigarrow \begin{bmatrix}1\\-1\end{bmatrix} \text{ is an eigenvalue for } 8 \\ A + 2I &= \begin{bmatrix}7 & -3\\ -7 & 3\end{bmatrix} \to \begin{bmatrix}7 & -3\\ 0 & 0\end{bmatrix} \rightsquigarrow \begin{bmatrix}3\\7\end{bmatrix} \text{ is an eigenvalue for } -2 \end{aligned}$$

Therefore $[T]_{\mathscr{B}}$ is diagonal for the basis

$$\mathscr{B} = \left\{\begin{bmatrix}1\\-1\end{bmatrix}, \begin{bmatrix}3\\7\end{bmatrix}\right\}$$

Problem 4. If A is similar to B, there exists an invertible matrix P such that $A = PBP^{-1}$. Then

$$A^2 = (PBP^{-1})(PBP^{-1}) = PBBP^{-1} = PB^2P^{-1}$$

Therefore, A^2 is similar to B^2.

Problem 5. If B is similar to A, then there exists an invertible matrix P such that

$$B = PAP^{-1} \Rightarrow A = P^{-1}BP$$

If C is similar to A, then there exists an invertible matrix Q such that

$$C = QAQ^{-1}$$

Combining the two equality, we get

$$C = QAQ^{-1} = QP^{-1}BPQ^{-1}$$

Since QP^{-1} is invertible and $PQ^{-1} = (QP^{-1})^{-1}$, we see that C is similar to B.

Problem 6. A is diagonalizable by definition means that A is similar to some diagonal matrix D. Since B is similar to A, by problem 23 above, B is similar to D. Hence B is diagoanliazble.