Worksheet 21	**Name:**	______________________
MATH 240 (Spring 2024)	**Section:**	0111(12PM-1PM) / 0121(1PM-2PM)
Apr 18th, 2024	**TA:**	Shin Eui Song

1. **Suggested Problems**

Problem 1 (5.5.2, 5.5.4). Let $\mathbb{C}^2$ act on the following matrices. Find the eigenvalues and a basis for each eigenspace in $\mathbb{C}^2$.

a)
$$\begin{bmatrix} -1 & -1 \\ 5 & -5 \end{bmatrix}$$

b)
$$\begin{bmatrix} -3 & -1 \\ 2 & -5 \end{bmatrix}$$

Problem 2 (6.1.5). Let $u = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Compute $\left(\frac{u \cdot v}{v \cdot v}\right) v$.

Problem 3 (6.1.13). Find the distance between $x = \begin{bmatrix} 10 \\ -3 \end{bmatrix}$ and $y = \begin{bmatrix} -1 \\ -5 \end{bmatrix}$.

Problem 4 (6.1.15, 6.1.16). Determine which pairs of vectors are orthogonal.

a) $a = \begin{bmatrix} 8 \\ -5 \end{bmatrix}, b = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$

b) $u = \begin{bmatrix} 12 \\ 3 \\ -5 \end{bmatrix}, v = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}$

Problem 5 (6.1.33). Let $v = \begin{bmatrix} a \\ b \end{bmatrix}$. Describe the set H of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ that are orthogonal to v. [Hint: Consider $v = 0$ and $v \neq 0$.]

Problem 6 (6.1.19). **(T/F)** $v \cdot v = ||v||^2$,

Problem 7 (6.1.20). **(T/F)** $u \cdot v - v \cdot u = 0$

Problem 8 (6.1.21). **(T/F)** If the distance from u to v equals the distance from u to $-v$, then u and v are orthogonal.

Problem 9 (6.1.23). **(T/F)** If vectors $v_1, \ldots, v_p$ span a subspace W and if x is orthogonal to each v_j for $j = 1, \ldots, p$, then x is in $W^\perp$.

Problem 1.

(a) The characteristic polynomial is

$$\det \begin{bmatrix} -1-\lambda & -1 \\ 5 & -5-\lambda \end{bmatrix} = \lambda^2 + 6\lambda + 5 + 5 = \lambda^2 + 6\lambda + 10$$

Therefore, the eigenvalues are

$$\lambda = \frac{-6 \pm \sqrt{36-40}}{2} = -3 \pm i$$

Since we have two distinct eigenvalues for 2×2 matrix, it is automatic that the eigenspace have dimension 1.

To find the eigenvector for $-3-i$, we need to look at

$$\begin{bmatrix} 2+i & -1 \\ 5 & -2+i \end{bmatrix} \to \begin{bmatrix} 2+i & -1 \\ 0 & 0 \end{bmatrix}$$

We know that the nullity is 1, so the dimension of the row space (which equals the dimension of the column space) is 1. Therefore, we automatically get the second row is a scalar multiple of the first row. To see directly, we have

$$(2-i)\begin{bmatrix} 2+i & -1 \end{bmatrix} = \begin{bmatrix} 5 & -2+i \end{bmatrix}$$

Therefore, an eigenvector for $-3-i$ is $\begin{bmatrix} 1 \\ 2+i \end{bmatrix}$ and $\left\{ \begin{bmatrix} 1 \\ 2+i \end{bmatrix} \right\}$ is the basis for the eigenspace for $-3-i$.

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$$\begin{bmatrix} 2-i & -1 \\ 5 & -2-i \end{bmatrix} \to \begin{bmatrix} 2-i & -1 \\ 0 & 0 \end{bmatrix}$$

Therefore, an eigenvector for $-3+i$ is $\begin{bmatrix} 1 \\ 2-i \end{bmatrix}$ and $\left\{ \begin{bmatrix} 1 \\ 2-i \end{bmatrix} \right\}$ is the basis for the eigenspace for $-3+i$.

(b) The characteristic polynomial is

$$\det \begin{bmatrix} -3-\lambda & -1 \\ 2 & -5-\lambda \end{bmatrix} = \lambda^2 + 8\lambda + 15 + 2 = \lambda^2 + 8\lambda + 17$$

The eigenvalues are

$$\lambda = \frac{-8 \pm \sqrt{64-68}}{2} = -4 \pm i$$

For $-4-i$,

$$\begin{bmatrix} 1+i & -1 \\ 2 & -1+i \end{bmatrix} \to \begin{bmatrix} 1+i & -1 \\ 0 & 0 \end{bmatrix}$$

By same reasoning as above, $\left\{ \begin{bmatrix} 1 \\ 1+i \end{bmatrix} \right\}$ is the basis for the eigenspace for $-4-i$.

For $-4+i$,

$$\begin{bmatrix} 1-i & -1 \\ 2 & -1-i \end{bmatrix} \to \begin{bmatrix} 1-i & -1 \\ 0 & 0 \end{bmatrix}$$

Therefore, $\left\{ \begin{bmatrix} 1 \\ 1-i \end{bmatrix} \right\}$ is the basis for the eigenspace for $-4-i$.

Problem 2.

$$u \cdot v = -2 + 6 = 4 \text{ and } v \cdot v = 4 + 9 = 13$$

Therefore,

$$\left(\frac{u\cdot v}{v\cdot v}\right)v = \frac{4}{13}\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}\frac{8}{13}\\ \frac{12}{13}\end{bmatrix}$$

Problem 3. The distance is

$$\text{dist}(x,y) = ||x-y|| = \left|\left|\begin{bmatrix}11\\2\end{bmatrix}\right|\right| = \sqrt{11^2+2^2} = \sqrt{125} = 5\sqrt{5}$$

Problem 4.

(a) $a\cdot b = -16+15 = -1 \neq 0$, not orthogonal

(b) $u\cdot v = 24-9-15 = 0$ orthogonal.

Problem 5.

(a) Suppose $v = 0$. Then for any vector $\begin{bmatrix}x\\y\end{bmatrix}$, $v\cdot\begin{bmatrix}x\\y\end{bmatrix} = 0$. Therefore,

$$H = \mathbb{R}^2.$$

(b) Suppose $v \neq 0$. Here we have two cases $a \neq 0$ or $b \neq 0$.

(i) Assume $a \neq 0$, and consider the equation

$$v\cdot\begin{bmatrix}x\\y\end{bmatrix} = ax+by = 0$$

Then $ax = -by \Rightarrow x = -\frac{b}{a}y$. Note here that we were able to divide by a because $a \neq 0$. Therefore,

$$H = \left\{\begin{bmatrix}-\frac{b}{a}y\\y\end{bmatrix}\middle|\, y\in\mathbb{R}\right\} = \text{Span}\left\{\begin{bmatrix}-\frac{b}{a}\\1\end{bmatrix}\right\}$$

(ii) Assume $b \neq 0$. By repeating the step (i) with the roles of a and b interchanged, we get

$$H = \left\{\begin{bmatrix}x\\-\frac{a}{b}x\end{bmatrix}\middle|\, y\in\mathbb{R}\right\} = \text{Span}\left\{\begin{bmatrix}1\\-\frac{a}{b}\end{bmatrix}\right\}$$

Problem 6. (True) See pg. 351.

Problem 7. (True) The dot product is commutative, i.e. $u\cdot v = v\cdot u$. Hence, $u\cdot v - v\cdot u = 0$.

Problem 8. (True) To paraphrase, we have

$$\begin{array}{rrcl} & ||u-v|| &=& ||u+v|| \\ \Rightarrow & ||u-v||^2 &=& ||u+v||^2 \\ \Rightarrow & (u-v)\cdot(u-v) &=& (u+v)\cdot(u+v) \\ \Rightarrow & ||u||^2 - 2(u\cdot v) + ||v||^2 &=& ||u||^2 + 2(u\cdot v) + ||v||^2 \\ \Rightarrow & -2(u\cdot v) &=& 2(u\cdot v) \\ \Rightarrow & 0 &=& 4(u\cdot v) \end{array}$$

Hence we have $u\cdot v = 0$, and u is orthogonal to v.

Problem 9. (True) Any element w in W is of the form

$$w = a_1v_1 + \cdots + a_pv_p$$

for some real numbers $a_1,\ldots,a_p\in\mathbb{R}$. Then

$$x\cdot w = x\cdot(a_1v_1+\cdots+a_pv_p) = a_1\underbrace{(x\cdot v_1)}_{=0} + \cdots + a_n\underbrace{(x\cdot v_p)}_{=0} = 0$$

Therefore x is orthogonal to any element w in W, and, by definition, $x\in W^\perp$.