Worksheet 22	**Name:**	______________________________
MATH 240 (Spring 2024)	**Section:**	0111(12PM-1PM) / 0121(1PM-2PM)
Apr 23th, 2024	**TA:**	Shin Eui Song

1. **Suggested Problems**

Problem 1 (6.2.8)**.** Show that $\{u_1, u_2\}$ is an orthogonal basis for $\mathbb{R}^2$. Then express x as a linear combination of the u's.

$$u_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} -2 \\ 6 \end{bmatrix}, \text{ and } \quad x = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

Problem 2 (6.2.13)**.** Let $y = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $u = \begin{bmatrix} 4 \\ -7 \end{bmatrix}$. Write y as the sum of two orthogonal vectors, one in $\mathrm{Span}\{u\}$ and one orthogonal to u.

Problem 3 (6.2.36)**.** Let U be an $n \times n$ orthogonal matrix. Show that the rows of U form an orthonormal basis of $\mathbb{R}^n$.

Problem 4 (6.2.37)**.** Let U and V be $n \times n$ orthogonal matrices. Explain why UV is an orthogonal matrix. [That is, explain why UV is invertible and its inverse is $(UV)^T$.]

Problem 5 (6.3.11)**.** Find the closest point to y in the subspace W spanned by v_1 and v_2 where

$$y = \begin{bmatrix} 3 \\ 1 \\ 5 \\ 1 \end{bmatrix}, \quad v_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

Problem 6 (6.3.19)**.** Let $u_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$, $u_2 = \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}$, and $u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Note that u_1 and u_2 are orthogonal but that u_3 is not orthogonal to u_1 or u_2. It can be shown that u_3 is not in the subspace W spanned by u_1 and u_2. Use this fact to construct a nonzero vector v in $\mathbb{R}^3$ that is orthogonal to u_1 and u_2.

Problem 1. We have $u_1 \cdot u_2 = -6 + 6 = 0$. Hence $\{u_1, u_2\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^2$. By Theorem 4 of Chapter 6, $\{u_1, u_2\}$ is a linear independent set with two vectors in $\mathbb{R}^2$. In particular, it is a basis of $\mathbb{R}^2$.

We want to find c_1 and c_2 such that $x = c_1u_1 + c_2u_2$. By Theorem 5 of Chapter 6, we have

$$c_1 = \frac{x \cdot u_1}{u_1 \cdot u_1} = \frac{-9}{10} \quad \text{and} \quad c_2 = \frac{x \cdot u_2}{u_2 \cdot u_2} = \frac{26}{40} = \frac{13}{20}.$$

To conclude,

$$x = -\frac{9}{10}u_1 + \frac{13}{20}u_2.$$

Problem 2. We first find a nonzero vector that is orthogonal to u. By inspection, we know that $v = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$ is orthogonal to u. $\{u, v\}$ is an orthogonal set with two vectors in $\mathbb{R}^2$, so it is an orthogonal basis. By the same arguments as of Problem 1,

$$c_1 = \frac{y \cdot u}{u \cdot u} = \frac{-13}{65} = -\frac{1}{5} \quad \text{and} \quad c_2 = \frac{y \cdot v}{v \cdot v} = \frac{26}{65} = \frac{2}{5}$$

Hence

$$y = -\frac{1}{5}u + \frac{2}{5}v.$$

Problem 3. U is orthogonal, then U^T is also orthogonal. This is because $(U^T)^T = U$ is the inverse of U^T. Then the set of columns of U^T, which is the set of rows of U, form an orthonormal basis of $\mathbb{R}^n$.

Problem 4. We have

$$(UV)^T UV = V^T U^T UV = V^T IV = V^T V = I$$

Therefore, UV is invertible, and $(UV)^T$ is the inverse of UV.

Problem 5. First, we need to find an orthonormal basis of W. Note that

$$v_1 \cdot v_2 = 3 - 1 - 1 - 1 = 0$$

Therefore, $\{u_1, u_2\}$ is an orthonormal basis. Then the projection $\hat{y}$ of y is given by

$$\hat{y} = c_1v_1 + c_2v_2$$

where

$$\begin{aligned} c_1 &= \frac{y \cdot v_1}{v_1 \cdot v_1} = \frac{6}{12} = \frac{1}{2} \\ c_2 &= \frac{y \cdot v_2}{v_2 \cdot v_2} = \frac{6}{4} = \frac{3}{2} \end{aligned}$$

Therefore

$$\hat{y} = \frac{1}{2}\begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix} + \frac{3}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

Problem 6. Let $\hat{u}_3$ be the projection of u_3 onto W. Then we have a decomposition $u_3 = \hat{u}_3 + z$ where z is orthogonal to any vector in W. Hence we compute $\hat{u}_3$ first. Similar to Problem 5, we have

$$\hat{u}_3 = c_1u_1 + c_2u_2$$

where

$$\begin{aligned} c_1 &= \frac{u_3 \cdot u_1}{u_1 \cdot u_1} = -\frac{2}{6} = -\frac{1}{3} \\ c_2 &= \frac{u_3 \cdot u_2}{u_2 \cdot u_2} = -\frac{2}{30} = -\frac{1}{15} \end{aligned}$$

Therefore

$$u_3 = -\frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + \frac{1}{15}\begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} + z$$

Solve for z, one should get

$$z = \begin{bmatrix} 0 \\ \frac{2}{5} \\ \frac{1}{5} \end{bmatrix}$$