Math 240 - Discussion
April 25th

Problem 1

Let

$$A = \begin{bmatrix} 1 & 7 \\ -4 & -7 \\ 0 & -4 \\ 1 & 1 \end{bmatrix}$$

and let $W = \mathrm{Col}(A)$.

1. Use the Gram-Schmidt process to produce an orthogonal basis for W.
2. Find a basis of $W^{\perp}$.

1. $A = \begin{bmatrix} 1 & 7 \\ -4 & -7 \\ 0 & -4 \\ 1 & 1 \end{bmatrix}$ and find orthogonal basis of W.

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$$c \begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ -7 \\ -4 \\ 1 \end{bmatrix} \quad \begin{array}{l} \Rightarrow \quad c \text{ cannot exist} \\ \Rightarrow \quad \text{the set of columns of } A \text{ is linearly independent} \end{array}$$

1. $A = \begin{bmatrix} 1 & 7 \\ -4 & -7 \\ 0 & -4 \\ 1 & 1 \end{bmatrix}$ and find orthogonal basis of W.

2.

$$v_1 = a_1 = \begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix}, \qquad \begin{aligned} v_2 &= a_2 - \frac{a_2 \cdot v_1}{v_1 \cdot v_1} v_1 \\ &= \begin{bmatrix} 7 \\ -7 \\ -4 \\ 1 \end{bmatrix} - \frac{7+28+0+1}{1+16+0+1} \begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 5 \\ 1 \\ -4 \\ -1 \end{bmatrix} \end{aligned}$$

1. $A = \begin{bmatrix} 1 & 7 \\ -4 & -7 \\ 0 & -4 \\ 1 & 1 \end{bmatrix}$ and find orthogonal basis of W.

$$\left\{ \begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ -4 \\ -1 \end{bmatrix} \right\}$$

2. $A = \begin{bmatrix} 1 & 7 \\ -4 & -7 \\ 0 & -4 \\ 1 & 1 \end{bmatrix}$ and find a basis for $W^{\perp}$.

We use the fact that $W^{\perp} = (\mathrm{Col}\ A)^{\perp} = \mathrm{Null}\ A^T$.

$$A^T = \begin{bmatrix} 1 & -4 & 0 & 1 \\ 7 & -7 & -4 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -4 & 0 & 1 \\ 0 & 21 & -4 & -6 \end{bmatrix}$$

Then the general solution is

$$\begin{bmatrix} 4x_2 - x_4 \\ \frac{4x_3+6x_4}{21} \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{16x_3+3x_4}{21} \\ \frac{4x_3+6x_4}{21} \\ x_3 \\ x_4 \end{bmatrix} \Rightarrow \left\{ \begin{bmatrix} \frac{16}{21} \\ \frac{4}{21} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{3}{21} \\ \frac{6}{21} \\ 0 \\ 1 \end{bmatrix} \right\}$$

Problem 2

Find an orthogonal basis for the column space of

$$A = \begin{bmatrix} 1 & 2 & 5 \\ -1 & 1 & -4 \\ -1 & 4 & -3 \\ 1 & -4 & 7 \\ 1 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 5 \\ -1 & 1 & -4 \\ -1 & 4 & -3 \\ 1 & -4 & 7 \\ 1 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 5 \\ 0 & 3 & 1 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{L.I.} \quad , v_1 = a_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 5 \\ -1 & 1 & -4 \\ -1 & 4 & -3 \\ 1 & -4 & 7 \\ 1 & 2 & 1 \end{bmatrix}, v_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{aligned} v_2 &= a_2 - \frac{a_2 \cdot v_1}{v_1 \cdot v_1} v_1 \\ &= \begin{bmatrix} 2 \\ 1 \\ 4 \\ -4 \\ 2 \end{bmatrix} - \frac{2-1-4-4+2}{5} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ 0 \\ 3 \\ -3 \\ 3 \end{bmatrix} \end{aligned}$$

$$A = \begin{bmatrix} 1 & 2 & 5 \\ -1 & 1 & -4 \\ -1 & 4 & -3 \\ 1 & -4 & 7 \\ 1 & 2 & 1 \end{bmatrix}, v_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 3 \\ 0 \\ 3 \\ -3 \\ 3 \end{bmatrix}$$

$$\begin{aligned} v_3 &= a_3 - \frac{a_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{a_3 \cdot v_2}{v_2 \cdot v_2} v_2 \\ &= \begin{bmatrix} 5 \\ -4 \\ -3 \\ 7 \\ 1 \end{bmatrix} - \frac{5+4+3+7+1}{5} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} - \frac{15+0-9-21+3}{36} \begin{bmatrix} 3 \\ 0 \\ 3 \\ -3 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 0 \\ 2 \\ 2 \\ -2 \end{bmatrix} \end{aligned}$$

Problem 3

The columns of Q were obtained by applying the Gram-Schmidt process to the columns of A. Find an upper triangular matrix R such that $A = QR$.

$$A = \begin{bmatrix} 5 & 9 \\ 1 & 7 \\ -3 & -5 \\ 1 & 5 \end{bmatrix}, \quad Q = \begin{bmatrix} 5/6 & -1/6 \\ 1/6 & 5/6 \\ -3/6 & 1/6 \\ 1/6 & 3/6 \end{bmatrix}$$

$$A = \begin{bmatrix} 5 & 9 \\ 1 & 7 \\ -3 & -5 \\ 1 & 5 \end{bmatrix}, \quad Q = \begin{bmatrix} 5/6 & -1/6 \\ 1/6 & 5/6 \\ -3/6 & 1/6 \\ 1/6 & 3/6 \end{bmatrix}$$

$$A = QR \Rightarrow Q^T A = Q^T QR \Rightarrow R = Q^T A$$

Therefore

$$R = \frac{1}{6} \begin{bmatrix} 5 & 1 & -3 & 1 \\ -1 & 5 & 1 & 3 \end{bmatrix} \begin{bmatrix} 5 & 9 \\ 1 & 7 \\ -3 & -5 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 12 \\ 0 & 6 \end{bmatrix}$$

Problem 4

Find a QR factorization of the matrix

$$A = \begin{bmatrix} 1 & 2 & 5 \\ -1 & 1 & -4 \\ -1 & 4 & -3 \\ 1 & -4 & 7 \\ 1 & 2 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 5 \\ -1 & 1 & -4 \\ -1 & 4 & -3 \\ 1 & -4 & 7 \\ 1 & 2 & 1 \end{bmatrix}, \quad \left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 3 \\ -3 \\ 3 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 2 \\ 2 \\ -2 \end{bmatrix} \right\}$$

The norm of each vectors are $\sqrt{5}, 6$, and 4. Therefore

$$Q = \begin{bmatrix} 1/\sqrt{5} & 1/2 & 1/2 \\ -1/\sqrt{5} & 0 & 0 \\ -1/\sqrt{5} & 1/2 & 1/2 \\ 1/\sqrt{5} & -1/2 & 1/2 \\ 1/\sqrt{5} & 1/2 & -1/2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 5 \\ -1 & 1 & -4 \\ -1 & 4 & -3 \\ 1 & -4 & 7 \\ 1 & 2 & 1 \end{bmatrix}, \quad \left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 3 \\ -3 \\ 3 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 2 \\ 2 \\ -2 \end{bmatrix} \right\}, Q = \begin{bmatrix} 1/\sqrt{5} & 1/2 & 1/2 \\ -1/\sqrt{5} & 0 & 0 \\ -1/\sqrt{5} & 1/2 & 1/2 \\ 1/\sqrt{5} & -1/2 & 1/2 \\ 1/\sqrt{5} & 1/2 & -1/2 \end{bmatrix}$$

Therefore,

$$R = Q^T A = \begin{bmatrix} 1/\sqrt{5} & -1/\sqrt{5} & -1/\sqrt{5} & 1/\sqrt{5} & 1/\sqrt{5} \\ 1/2 & 0 & 1/2 & -1/2 & 1/2 \\ 1/2 & 0 & 1/2 & 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 5 \\ -1 & 1 & -4 \\ -1 & 4 & -3 \\ 1 & -4 & 7 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 5/\sqrt{5} & -5/\sqrt{5} & 20/\sqrt{5} \\ 0 & 6 & -2 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{5} & -\sqrt{5} & 4\sqrt{5} \\ 0 & 6 & -2 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 5 \\ -1 & 1 & -4 \\ -1 & 4 & -3 \\ 1 & -4 & 7 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} & 1/2 & 1/2 \\ -1/\sqrt{5} & 0 & 0 \\ -1/\sqrt{5} & 1/2 & 1/2 \\ 1/\sqrt{5} & -1/2 & 1/2 \\ 1/\sqrt{5} & 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} \sqrt{5} & -\sqrt{5} & 4\sqrt{5} \\ 0 & 6 & -2 \\ 0 & 0 & 4 \end{bmatrix}$$

Problem 5

1. If $\{v_1, v_2, v_3\}$ is an orthogonal basis for W, then multiplying v_3 by a scalar c gives a new orthogonal basis $\{v_1, v_2, cv_3\}$.

When $c = 0$, the set $\{v_1, v_2, cv_3\}$ is no longer linearly independent. **False**. However, if the problem said $c \neq 0$, then it is **True**.

Problem 5

2. If $W = \mathrm{Span}\{x_1, x_2, x_3\}$ with $\{x_1, x_2, x_3\}$ linearly independent, and if $\{v_1, v_2, v_3\}$ is an orthogonal set in W, then $\{v_1, v_2, v_3\}$ is a basis of W.

Since $\dim\ W = 3$, and any orthogonal set is linearly indendent. $\{v_1, v_2, v_3\}$ has three vectors. **True**.

Problem 5

3. The Gram-Schmidt process produces from a linearly independent set $\{x_1, \ldots, x_p\}$ an orthogonal set $\{v_1, \ldots, v_p\}$ with the property that for each k, the vectors $v_1, \ldots, v_k$ span the same subspace as that spanned by $x_1, \ldots, x_k$.

This is Theorem 11 from Chapter 6. **True**.

Problem 5

4. If x is not in a subspace W, then $x - \mathrm{proj}_W x$ is not zero.

$x - \mathrm{Proj}_W x = 0$ if and only if $x \in W$. **True**.

Problem 5

5. If $A = QR$, where Q has orthonormal columns, then $R = Q^T A$.

$A = QR \Rightarrow Q^T A = Q^T QR \Rightarrow R = Q^T A$. **True**.

Problem 5

6. In a QR factorization, say $A = QR$ (when A has linearly independent columns), the columns of Q form an orthonormal basis for the column space of A.

Q is orthogonal, so its columns form an orthonormal basis. **True**.