Worksheet 3 **Name:** ______________________________
MATH 240 (Spring 2024) **Section:** 0111(12PM-1PM) / 0121(1PM-2PM)
Feb 6, 2024 **TA:** Shin Eui Song

1. Suggested Problems

Problem 1 (1.3.5, 1.3.6)**.** Write a system of equations that is equivalent to the given vector equation.

(a)

$$x_1 \begin{bmatrix} 6 \\ -1 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -7 \\ -5 \end{bmatrix}$$

(b)

$$x_1 \begin{bmatrix} -2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 8 \\ 5 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Problem 2 (1.3.12)**.** Determine if $\boldsymbol{b}$ is a linear combination of $\boldsymbol{a}_1, \boldsymbol{a}_2$, and $\boldsymbol{a}_3$.

$$\boldsymbol{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \quad \boldsymbol{a}_2 = \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix}, \quad \boldsymbol{a}_3 = \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} -5 \\ 11 \\ -7 \end{bmatrix}$$

Problem 3 (1.3.18)**.** Let $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$, $\boldsymbol{v}_2 = \begin{bmatrix} -3 \\ 1 \\ 8 \end{bmatrix}$, and $\boldsymbol{y} = \begin{bmatrix} h \\ -5 \\ -3 \end{bmatrix}$. For what value(s) of h is $\boldsymbol{y}$ in the plane generated by $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$?

2. Additional Problems

Problem 4. Determine which spanning set is a point, a line, a plane, or a space (3-dimensional space).

a)

$$\text{Span}\left\{ \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \end{bmatrix} \right\}$$

b)

$$\text{Span}\left\{ \begin{bmatrix} 0 \\ -4 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} \right\}$$

c)

$$\text{Span}\left\{ \begin{bmatrix} \pi \\ 0 \\ \sqrt{2} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} \right\}$$

d)

$$\text{Span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

e)

$$\text{Span}\left\{ \begin{bmatrix} 2.5 \\ 1.5 \end{bmatrix}, \begin{bmatrix} -10 \\ -6 \end{bmatrix} \right\}$$

Problem 5. Prove that $\text{Span}\{\mathbf{u}, \mathbf{v}\} = \text{Span}\{\mathbf{u}, \mathbf{u} + \mathbf{v}\}$ where $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$.

Problem 6. Let $\mathbf{u} = \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$ be a column vector with two entries. Can you geometrically describe what happens to $\mathbf{u}$ after the matrix muliplication

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

(**Hint**: Plug in a vectors such that $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to see the pattern.)

Problem 1.

(a)

$$\begin{aligned} 6x_1 - 3x_2 &= 1 \\ -x_1 + 4x_2 &= -7 \\ 5x_1 \quad\quad &= -5 \end{aligned}$$

(b)

$$\begin{aligned} -2x_1 + 8x_2 + x_3 &= 0 \\ 3x_1 + 5x_2 - 6x_3 &= 0 \end{aligned}$$

Problem 2. We just need to solve the linear system associated to the matrix

$$\begin{bmatrix} 1 & 0 & 2 & -5 \\ -2 & 5 & 0 & 11 \\ 2 & 5 & 8 & -7 \end{bmatrix} \xrightarrow[R_3 = R_3 - 2R_1]{R_2 = R_2 + 2R_1} \begin{bmatrix} 1 & 0 & 2 & -5 \\ 0 & 5 & 4 & 1 \\ 0 & 5 & 4 & 3 \end{bmatrix}$$

We see that the linear system is inconsistent, so $\boldsymbol{b}$ is not a linear combination of $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, and $\boldsymbol{a}_3$.

Problem 3. We solve $\boldsymbol{v}_1 x_1 + \boldsymbol{v}_2 x_2 = \boldsymbol{y}$. The second entries tell us that $x_2 = -5$. Then by back-substitution, $-2x_1 + 8x_2 = -3 \Rightarrow -2x_1 = 37 \Rightarrow x_1 = -\frac{37}{2}$. Therefore, one must have

$$h = x_1 - 3x_2 = -\frac{37}{2} - 3 \cdot -5 = -\frac{37}{2} + \frac{30}{2} = -\frac{7}{2}.$$

Alternatively, we can row reduce the associated matrix.

$$\begin{bmatrix} 1 & -3 & h \\ 0 & 1 & -5 \\ -2 & 8 & -3 \end{bmatrix} \xrightarrow{R_3 = R_3 - 8R_2} \begin{bmatrix} 1 & -3 & h \\ 0 & 1 & -5 \\ -2 & 0 & 37 \end{bmatrix} \xrightarrow{R_3 = -\frac{1}{2}R_3} \begin{bmatrix} 1 & -3 & h \\ 0 & 1 & -5 \\ 1 & 0 & -\frac{37}{2} \end{bmatrix}$$

$$\xrightarrow{R_1 = R_1 - R_3 + 3R_2} \begin{bmatrix} 0 & 0 & h + \frac{37}{2} - 15 \\ 0 & 1 & -5 \\ 1 & 0 & -\frac{37}{2} \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & -\frac{37}{2} \\ 0 & 1 & -5 \\ 0 & 0 & h + \frac{7}{2} \end{bmatrix}$$

Then for the linear system to be consistent, we need $h + \frac{7}{2} = 0$, i.e. $h = -\frac{7}{2}$.

Problem 4. Do not worry too much about this question. The goal of the problem was to show you that a span of two vectors is not always a plane, and a span of three vectors is not always a space. You don't have to understand this completely.

(a) The set is spanned by two vectors in $\mathbb{R}^2$ that is not scalar multiples of each other. Therefore, it spans the plane $\mathbb{R}^2$.

(b)

$$\begin{bmatrix} 0 \\ -4 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$$

One can show that

$$\text{Span}\left\{ \begin{bmatrix} 0 \\ -4 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} \right\} = \text{Span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} \right\} = \text{Span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Then now it is easy that spanning set is a plane in $\mathbb{R}^3$.

(c) We first show that

$$\text{Span}\left\{\begin{bmatrix}\pi\\0\\\sqrt{2}\end{bmatrix},\begin{bmatrix}0\\0\\5\end{bmatrix}\right\} = \text{Span}\left\{\begin{bmatrix}\pi\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\5\end{bmatrix}\right\}$$

This is same as showing that

$$\begin{bmatrix}\pi\\0\\\sqrt{2}\end{bmatrix},\begin{bmatrix}0\\0\\5\end{bmatrix} \in \text{Span}\left\{\begin{bmatrix}\pi\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\5\end{bmatrix}\right\} \text{ and } \begin{bmatrix}\pi\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\5\end{bmatrix} \in \text{Span}\left\{\begin{bmatrix}\pi\\0\\\sqrt{2}\end{bmatrix},\begin{bmatrix}0\\0\\5\end{bmatrix}\right\}$$

If you don't see this, don't worry about it yet. We will cover this in more detail later in the course. Next, we have

$$\text{Span}\left\{\begin{bmatrix}\pi\\0\\\sqrt{2}\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\5\end{bmatrix}\right\} = \text{Span}\left\{\begin{bmatrix}\pi\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\5\end{bmatrix}\right\} = \text{Span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$$

where the last spanning set is the entire space $\mathbb{R}^3$.

(d)

$$\text{Span}\left\{\begin{bmatrix}0\\0\\0\end{bmatrix}\right\} = \left\{\mathbf{b} \in \mathbb{R}^3 \;\middle|\; a\begin{bmatrix}0\\0\\0\end{bmatrix} = \mathbf{b}\right\} = \left\{\begin{bmatrix}0\\0\\0\end{bmatrix}\right\}$$

The last equality follows from the fact that

$$a\begin{bmatrix}0\\0\\0\end{bmatrix} = \begin{bmatrix}a\cdot 0\\a\cdot 0\\a\cdot 0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

(e) Since

$$-4\begin{bmatrix}2.5\\1.5\end{bmatrix} = \begin{bmatrix}-10\\-6\end{bmatrix}$$

We have

$$\text{Span}\left\{\begin{bmatrix}2.5\\1.5\end{bmatrix},\begin{bmatrix}-10\\-6\end{bmatrix}\right\} = \text{Span}\left\{\begin{bmatrix}2.5\\1.5\end{bmatrix}\right\}$$

A span of a nonzero vector is a line.

Problem 5. To show that two sets are the same, we show two inclusions $\text{Span}\{\mathbf{u},\mathbf{u}+\mathbf{v}\} \subset \text{Span}\{\mathbf{u},\mathbf{v}\}$ and $\text{Span}\{\mathbf{u},\mathbf{v}\} \subset \text{Span}\{\mathbf{u},\mathbf{u}+\mathbf{v}\}$.

Let $a_1\mathbf{u} + a_2(\mathbf{u}+\mathbf{v}) \in \text{Span}\{\mathbf{u},\mathbf{u}+\mathbf{v}\}$. Then

$$a_1\mathbf{u} + a_2(\mathbf{u}+\mathbf{v}) = (a_1+a_2)\mathbf{u} + a_2\mathbf{v} \in \text{Span}\{\mathbf{u},\mathbf{v}\}$$

This shows that $\text{Span}\,\{\mathbf{u},\mathbf{u}+\mathbf{v}\} \subset \text{Span}\,\{\mathbf{u},\mathbf{v}\}$.

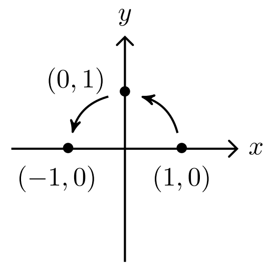
Conversely, let $a_1\mathbf{u} + a_2\mathbf{v} \in \text{Span}\,\{\mathbf{u},\mathbf{v}\}$. Then

$$a_1\mathbf{u} + a_2\mathbf{v} = (a_1-a_2)\mathbf{u} + a_2(\mathbf{u}+\mathbf{v}) \in \text{Span}\,\{\mathbf{u},\mathbf{u}+\mathbf{v}\}.$$

This shows that $\text{Span}\,\{\mathbf{u},\mathbf{v}\} \subset \text{Span}\,\{\mathbf{u},\mathbf{u}+\mathbf{v}\}$.

Problem 6. We answer by observation. This is NOT a complete rigorous answer.

$$\begin{bmatrix}0&-1\\1&0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix} \text{ and } \begin{bmatrix}0&-1\\1&0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-1\\0\end{bmatrix}$$



Therefore, we observed that multiplying the vector by a matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is same as rotating 90° counterclockwise.