Worksheet 4	**Name:**	______________________
MATH 240 (Spring 2024)	**Section:**	0111(12PM-1PM) / 0121(1PM-2PM)
Feb 8, 2024	**TA:**	Shin Eui Song

1. Suggested Problems

Problem 1 (1.4.11)**.** Given A and $\mathbf{b}$, write the augmented matrix for the linear system that corresponds to the matrix equation $A\mathbf{x} = \mathbf{b}$. Then solve the system and write the solution as a vector.

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 5 \\ -2 & -4 & -3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -2 \\ 2 \\ 9 \end{bmatrix}$$

Problem 2 (1.4.17)**.** How many rows of

$$A = \begin{bmatrix} 1 & 3 & 0 & 3 \\ -1 & -1 & -1 & 1 \\ 0 & -4 & 2 & -8 \\ 2 & 0 & 3 & -1 \end{bmatrix}$$

contains a pivot position? Does the equation $A\mathbf{x} = \mathbf{b}$ have a solution for each $\mathbf{b}$ in $\mathbb{R}^4$?

Problem 3 (1.4.28)**.** (T/F) If A is an $m \times n$ matrix whose columns do not span $\mathbb{R}^m$, then the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent for some $\mathbf{b}$ in $\mathbb{R}^m$.

2. Additional Problems: Mathematical Logic

Problem 4. Using the contrapositive, prove that "For all $x, y \in \mathbb{N}$, if xy is an even number, then x is even or y is even".

Problem 5. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. What is the negation of the proposition "For all $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$ for all $x \in \mathbb{R}$"?

Problem 6. Determine whether or not the following proposition is true. "For all 4×6 matrix A that is an augmented matrix of a linear system, if A has a 4 pivot columns, then A is consistent".

Problem 1. The augmented matrix is

$$\begin{bmatrix} 1 & 2 & 4 & -2 \\ 0 & 1 & 5 & 2 \\ -2 & -4 & -3 & 9 \end{bmatrix} \xrightarrow{R_3=R_3+2R1} \begin{bmatrix} 1 & 2 & 4 & -2 \\ 0 & 1 & 5 & 2 \\ 0 & 0 & 5 & 5 \end{bmatrix}$$

The third row tells us that $x_3 = 1$. The second row tells us that $x_2 + 5 = 2 \Rightarrow x_2 = -3$. Finally, the first row tells us that $x_1 - 6 + 4 = -2 \Rightarrow x_1 = 0$. This process is called back-substitution. In vector form, we have

$$\mathbf{x} = \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}$$

Problem 2.

$$\begin{bmatrix} 1 & 3 & 0 & 3 \\ -1 & -1 & -1 & 1 \\ 0 & -4 & 2 & -8 \\ 2 & 0 & 3 & -1 \end{bmatrix} \xrightarrow[R_4=R_4-2R_1]{R_2=R_2+R_1} \begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 2 & -1 & 4 \\ 0 & -4 & 2 & -8 \\ 0 & -6 & 3 & -7 \end{bmatrix} \xrightarrow[R_4=R_4+3R_2]{R_3=R_3+2R_2} \begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 2 & -1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

By switching row 3 and row 4, we see that the matrix has three pivot positions. Now we claim that the equation $A\mathbf{x} = \mathbf{b}$ does not have a solution for each $\mathbf{b}$ in $\mathbb{R}^4$. The negation of this proposition is "there exists $\mathbf{b}$ in $\mathbb{R}^m$ such that $A\mathbf{x} = \mathbf{b}$ does not have a solution." Consider

$$\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

The third row of the augmented matrix is $\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ which will make the system inconsistent.

Problem 3. Let $A = [\mathbf{a}_1 \cdots \mathbf{a}_n]$ a matrix whose columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$ do not span $\mathbb{R}^m$. Then by definition of span, there exists a $\mathbf{b} \in \mathbb{R}^m$ such that $\mathbf{b}$ cannot be written as a linear combination of $\mathbf{a}_1, \ldots, \mathbf{a}_n$. Then this is (logically) equivalent to saying that $x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$ does not have a solution. In matrix form, this is same as saying that $A\mathbf{x} = \mathbf{b}$ does not have a solution.

One can also note that the statement is the contrapositive item a $\Rightarrow$ item c of Theorem 4 in page 39 of the textbook. Hence the statement is **true**.

Problem 4. The contrapositive states that for all $x, y \in \mathbb{N}$, if x and y are odd, then xy is an odd number. Let $x = 2a + 1$ and $y = 2b + 1$ be odd numbers. Then $xy = (2a + 1)(2b + 1) = 4ab + 2a + 2b + 1 = 2(2ab + 2 + b) + 1$. Therefore, xy is odd and we have proved the contrapositive.

Problem 5. There exists $\varepsilon > 0$ such that for all $\delta > 0$, there exists an $x \in \mathbb{R}$ with $|x - a| < \delta$ and $|f(x) - f(a)| \geq \varepsilon$.

Problem 6. We claim that the proposition is false. To show this, we prove that the negation of the proposition is true. The negation is "There exists 4×6 matrix A that is an augmented matrix of linear system where A has 4 pivot columns and A is inconsistent". The matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

is inconsistent because the last column of the **augmented matrix** is a pivot position.