Worksheet 5	**Name:**	______________________
MATH 240 (Spring 2024)	**Section:**	0111(12PM-1PM) / 0121(1PM-2PM)
Feb 13, 2024	**TA:**	Shin Eui Song

1. **Suggested Problems**

Problem 1 (1.5.11)**.** Describe all solutions of $A\mathbf{x} = 0$ in parametric vector form, where A is row equivalent to

$$\begin{bmatrix} 1 & -4 & -2 & 0 & 3 & -5 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Problem 2 (1.5.27)**.** (T/F) A homogeneous equation is always consistent.

Problem 3 (1.5.36)**.** (T/F) The solution set of $A\mathbf{x} = \mathbf{b}$ is obtained by translating the solution set of $A\mathbf{x} = 0$.

Problem 4 (1.5.48)**.** Construct a 3×3 nonzero matrix A such that the vector $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ is a solution of $A\mathbf{x} = \mathbf{0}$.

Problem 5 (1.7.4)**.** Determine if the vectors are linearly independent. Justify your answer.

$$\begin{bmatrix} -1 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 8 \end{bmatrix}$$

Problem 6 (1.7.6)**.** Determine if the columns of the matrix form a linearly independent set. Justify your answer.

$$\begin{bmatrix} -4 & -3 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & 3 \\ 5 & 4 & 6 \end{bmatrix}$$

2. **Additional Problems**

Problem 7. Let $\mathbf{v} \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Show that $c\mathbf{v} \neq 0$ if and only if $c \neq 0$ and $\mathbf{v} \neq 0$.

Problem 8. Prove that $\{\mathbf{v}_1, \mathbf{v}_2\}$ are linearly dependent if and only if $\mathbf{v}_1$ and $\mathbf{v}_2$ are scalar multiples of each other.

Problem 9. Find a set $\{\mathbf{v}_1, \dots, \mathbf{v}_3\}$ of vectors in $\mathbb{R}^4$ such that $\mathbf{v}_2$ is not a linear combination of $\mathbf{v}_1$, but the set is linearly dependent.

Problem 10 (Challenge)**.** Let A be an $m \times n$ matrix,

$$A\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ has } \textbf{no solutions} \text{ and } A\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ has } \textbf{exactly one solution}.$$

(a) Show that $A\mathbf{x} = 0$ has only one solution.

(b) Can you determine m, n, and the number of pivot points of A?

(c) Write down a matrix A that fits the description in part (a).

Problem 1. The pivot positions are colored blue.

$$\begin{bmatrix} \mathbf{1} & -4 & -2 & 0 & 3 & -5 \\ 0 & 0 & \mathbf{1} & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & \mathbf{1} & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then the general solution as in Section 1.3 is given by

$$\begin{cases} x_1 = & 4x_2 + 2x_3 - 3x_5 + 5x_6 \\ x_2 & \text{free} \\ x_3 = & x_6 \\ x_4 & \text{free} \\ x_5 = & 4x_6 \\ x_6 & \text{free} \end{cases}$$

We rewrite x_1 in terms of only the free variables x_2, x_4, and x_6. In other words,

$$x_1 = 4x_2 + 2x_6 - 12x_6 + 5x_6 = 4x_2 - 5x_6$$

Then, in parametric vector form, we have

$$\mathbf{x} = \begin{bmatrix} 4x_2 - 5x_6 \\ x_2 \\ x_6 \\ x_4 \\ 4x_6 \\ x_6 \end{bmatrix} = x_2 \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 4 \\ 1 \end{bmatrix}$$

Problem 2. This is **true** as the zero vector is always a solution of a homogeneous equation $A\mathbf{x} = 0$. Remember that the zero vector is called the trivial solution of $A\mathbf{x} = 0$.

Problem 3. (There is subtlety!) If $A\mathbf{x} = \mathbf{b}$ is consistent, then this is *true* exactly Theorem 6 in page 49 and it is true. We can directly prove this as follows. Fix a solution $\mathbf{p}$ of $A\mathbf{x} + \mathbf{b}$. We would like to prove the equality

$$\{\mathbf{v} \mid \mathbf{v} \text{ is a solution of } A\mathbf{x} = \mathbf{b}\} = \{\mathbf{p} + \mathbf{v}_h \mid \mathbf{v}_h \text{ is a solution of } A\mathbf{x} = 0\}$$

First, consider a vector of the form $\mathbf{p} + \mathbf{v}_h$. Then $A(\mathbf{p} + \mathbf{v}_h) = A\mathbf{p} + A\mathbf{v}_h = A\mathbf{p} = \mathbf{b}$. Hence $\mathbf{p} + \mathbf{v}_h$ is a solution of $A\mathbf{x} = \mathbf{b}$ showing the inclusion

$$\{\mathbf{v} \mid \mathbf{v} \text{ is a solution of } A\mathbf{x} = \mathbf{b}\} \supset \{\mathbf{p} + \mathbf{v}_h \mid \mathbf{v}_h \text{ is a solution of } A\mathbf{x} = 0\}$$

Conversely, let $\mathbf{v}$ be a solution of $A\mathbf{x} = \mathbf{b}$. Then $A(\mathbf{v} - \mathbf{p}) = A\mathbf{v} - A\mathbf{p} = \mathbf{b} - \mathbf{b} = 0$, so $\mathbf{v} - \mathbf{p}$ is a solution of $A\mathbf{x} = \mathbf{b}$. Call it $\mathbf{v}_h := \mathbf{v} - \mathbf{p}$. Then since $\mathbf{v} = \mathbf{p} + (\mathbf{v} - \mathbf{p}) = \mathbf{w} + \mathbf{v}_h$, we have

$$\{\mathbf{v} \mid \mathbf{v} \text{ is a solution of } A\mathbf{x} = \mathbf{b}\} \subset \{\mathbf{p} + \mathbf{v}_h \mid \mathbf{v}_h \text{ is a solution of } A\mathbf{x} = 0\}$$

which shows the desired equality.

If $A\mathbf{x} = \mathbf{b}$ is inconsistent, this may be a *false* statement. Namely, the solution set of $A\mathbf{x} = \mathbf{b}$ is empty. However, any translate of a solution set of $A\mathbf{x} = 0$ is never empty because the solution set of $A\mathbf{x} = 0$ has at least one element, the trivial solution.

Problem 4. We can write an arbitrary 3×3 matrix as follows

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \Rightarrow \quad A \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} - 2a_{12} + a_{13} \\ a_{21} - 2a_{22} + a_{23} \\ a_{31} - 2a_{32} + a_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We solve this row-by-row. Namely, find a_{11}, a_{12}, a_{13} such that $a_{11} - 2a_{12} + a_{13} = 0$. The $a_{11} = a_{12} = a_{13} = 1$ works! By the same reasoning for other rows, the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

works.

Problem 5. Since

$$-2 \begin{bmatrix} -1 \\ 4 \end{bmatrix} + \begin{bmatrix} -2 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

the two vectors are linearly dependent.

Problem 6. To determine linear dependence, we have to figure out if the homogeneous equation

$$\begin{bmatrix} -4 & -3 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & 3 \\ 5 & 4 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

has a nontrivial solution.

The matrix is row reduced to

$$\begin{bmatrix} -4 & -3 & 0 & 0 \\ 0 & -1 & 4 & 0 \\ 1 & 0 & 3 & 0 \\ 5 & 4 & 6 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 \\ 1 & 0 & 3 & 0 \\ 0 & 4 & -9 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & -1 & 4 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there are three pivots, the only solution is $x_1 = x_2 = x_3 = 0$. In other words, the columns of the matrix is linearly independent.

Problem 7. We use the following property of real numbers. Let a and b in $\mathbb{R}$, then $ab = 0$ if and only if $a = 0$ or $b = 0$. Let $\mathbf{v} \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Then $c\mathbf{v} = \begin{bmatrix} cv_1 \\ \vdots \\ cv_n \end{bmatrix}$. Then $c\mathbf{v} = 0$ if and only if $cv_i = 0$ for all i if and only if $c = 0$ and $v_i = 0$ for all i if and only if $c = 0$ and $\mathbf{v} = 0$. This proves the contrapositive of the statement.

Problem 8. Let $\mathbf{v}_1$ be a scalar multiple of $\mathbf{v}_2$, i.e. $\mathbf{v}_1 = c\mathbf{v}_2$. If $c = 0$, then $\mathbf{v}_1$ is a zero vector, then $a\mathbf{v}_1 + 0\mathbf{v}_2 = 0$ with any nonzero $a \neq 0$ gives us a linear dependence relation. If $c \neq 0$, then $\mathbf{v}_1 - c\mathbf{v}_2 = 0$ gives us a linear dependence relation, i.e. $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent. Conversely, if $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent. Then we have $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = 0$ with not both c_1 and c_2 zero.

(a) $c_1 \neq 0$, then $\mathbf{v}_1 = -\frac{c_2}{c_1}\mathbf{v}_2$,

(b) $c_2 \neq 0$, then $\mathbf{v}_2 = -\frac{c_1}{c_2}\mathbf{v}_1$.

Problem 9. Choose any $\mathbf{v}_1$. Choose $\mathbf{v}_2$ such that $\mathbf{v}_2$ is not a scalar multiple of $\mathbf{v}_1$. Then choose $\mathbf{v}_3$ so that $\mathbf{v}_3$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. For example,

$$\left\{ \underbrace{\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}}_{=\mathbf{v}_1}, \underbrace{\begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}}_{=\mathbf{v}_2}, \underbrace{\begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}}_{=\mathbf{v}_3} \right\}$$

Here we chose $\mathbf{v}_3 = \mathbf{v}_1 + 2\mathbf{v}_2$.

Problem 10.

(a) The solution set of $A\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is of the form $\mathbf{p} + \mathbf{v}_h$ where $\mathbf{p}$ is a solution to the equation and $\mathbf{v}_h$ is the solution to $A\mathbf{x} = 0$. If the equation $A\mathbf{x} = 0$ has infinitely many solutions, $A\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ would have infinitely many solutions. We proved the contrapositve.

(b) Since the vector on the right-hand side is in $\mathbb{R}^3$, in order for the matrix multiplication to make sense, $m = 3$. As $A\mathbf{x} = 0$ has only one solution, we have $n \leq 3$. This is because if $n \geq 4$, you will have a free variable which will imply that $A\mathbf{x} = 0$ has infinitely many solutions. If $n = 3$, $A\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ having exactly one solution would mean that, A would have three pivot columns. However, this contradicts the fact that $A\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ has no solutions by Theorem 4 in page 39. In particular, $n = 1$ or $n = 2$.

(c) For $n = 1$, $A = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ works. For $n = 2$, $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ works. The column vectors are chosen so that the third coordinates are zero so that the column vectors cannot span $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and $\mathbf{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is a solution to $A\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.