Worksheet 6	**Name:** ____________________________
MATH 240 (Spring 2024)	**Section:** 0111(12PM-1PM) / 0121(1PM-2PM)
Feb 15, 2024	**TA:** Shin Eui Song

1. **Suggested Problems**

Problem 1 (1.7.25)**.** (T/F) The columns of any 4×5 matrix are linearly dependent.

Problem 2 (1.7.37)**.** Given

$$A = \begin{bmatrix} 2 & 3 & 5 \\ -5 & 1 & -4 \\ -3 & -1 & -4 \\ 1 & 0 & 1 \end{bmatrix}$$

observe that the third column is the sum of the first two columns. Find a nontrivial solution of $A\mathbf{x} = 0$.

For problem 3 and 4, determine whether each statement is true. If true, give justifications. If false, show a specific example (called **counterexample**) to show that the statement is not true. Below F-C means that if you claim that the statement is false, you need to provide a counterexample.

Problem 3 (1.7.39)**.** (T/F-C) If $\mathbf{v}_1, \dots, \mathbf{v}_4$ are in $\mathbb{R}^4$ and $\mathbf{v}_3 = 2\mathbf{v}_1 + \mathbf{v}_2$, then $\{\mathbf{v}_1, \dots, \mathbf{v}_4\}$ is linearly dependent.

Problem 4 (1.7.40)**.** (T/F-C) If $\mathbf{v}_1, \dots, \mathbf{v}_4$ are in $\mathbb{R}^4$ and $\mathbf{v}_3 = 0$, then $\{\mathbf{v}_1, \dots, \mathbf{v}_4\}$ is linearly dependent.

Problem 5 (1.8.11)**.** Let

$$\mathbf{b} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \text{ and } A = \begin{bmatrix} 1 & -4 & 7 & -5 \\ 0 & 1 & -4 & 3 \\ 2 & -6 & 6 & -4 \end{bmatrix}$$

Is $\mathbf{b}$ in the range of the linear tranformation $\mathbf{x} \mapsto A\mathbf{x}$? Why or why not?

Problem 6 (1.8.13)**.** Use a rectangular coordinate system to plot

$$\mathbf{u} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

and their images under the transformation

$$T(\mathbf{x}) = \begin{bmatrix} .5 & 0 \\ 0 & .5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Problem 7 (1.8.19)**.** Let

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{y}_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \text{ and } \mathbf{y}_2 = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$$

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that maps $\mathbf{e}_1$ into $\mathbf{y}_1$ and maps $\mathbf{e}_2$ into $\mathbf{y}_2$. Find the images of $\begin{bmatrix} 5 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

2. **Additional Problems**

Problem 8. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be an **affine transformation**, i.e. defined by

$$\mathbf{x} \mapsto A\mathbf{x} + \mathbf{b}$$

where A is an $m \times n$ matrix and $\mathbf{b} \in \mathbb{R}^m$. Show that T is linear if and only if $\mathbf{b} = 0$.

Problem 9.

(a) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation defined by

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}7\\-3\end{bmatrix} \text{ and } T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}-1\\6\end{bmatrix}$$

Show that you can always find a matrix A such that $T(\mathbf{x}) = A\mathbf{x}$.

(b) (Challenge) Can you generalize (a) to an arbitrary linear transformation? In other words, let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Find a matrix A such that $T(\mathbf{x}) = A\mathbf{x}$. What is the size of A?

Problem 10. (Challenge) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation defined by $\mathbf{x} \mapsto A\mathbf{x}$, and let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation defined by $\mathbf{x} \mapsto B\mathbf{x}$ where

$$A = \begin{bmatrix}a_{11} & a_{12}\\a_{21} & a_{22}\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}b_{11} & b_{12}\\b_{21} & b_{22}\end{bmatrix}$$

Without using matrix multiplication and using similar strategy to Problem 9, show that the composition $(T \circ S)(\mathbf{x}) = T(S(\mathbf{x})) = C\mathbf{x}$ where

$$C = \begin{bmatrix}a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22}\\a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22}\end{bmatrix}$$

Worksheet 6 Solution
MATH 240 (Spring 2024)

Problem 1. We have 5 vectors in $\mathbb{R}^4$. Since there are more vectors than the entries, they are linearly dependent by theorem 8 in page 63.

Problem 2. If we write the last sentence with mathematical expression , we have

$$\begin{bmatrix} 2 \\ -5 \\ -3 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \\ -4 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 \\ -5 \\ -3 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \\ -1 \\ 0 \end{bmatrix} - \begin{bmatrix} 5 \\ -4 \\ -4 \\ 1 \end{bmatrix} = 0$$

Then $x_1 = 1$, $x_2 = 1$, and $x_3 = -1$ is the solution to the vector equation. Therefore

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

is a solution to the corresponding matrix equation.

Problem 3. This is true because,

$$2\mathbf{v}_1 + \mathbf{v}_2 - 3\mathbf{v}_3 + 0\mathbf{v}_4 = 0$$

Problem 4. This is true because

$$0\mathbf{v}_1 + 0\mathbf{v}_2 + \mathbf{v}_3 + 0\mathbf{v}_4 = 0$$

Problem 5.

$$\begin{bmatrix} 1 & -4 & 7 & -5 & 1 \\ 0 & 1 & -4 & 3 & 1 \\ 2 & -6 & 6 & -4 & 0 \end{bmatrix} \xrightarrow{R_3 = R_3 - 2R_1} \begin{bmatrix} 1 & -4 & 7 & -5 & 1 \\ 0 & 1 & -4 & 3 & 1 \\ 0 & 2 & -8 & 6 & 2 \end{bmatrix}$$

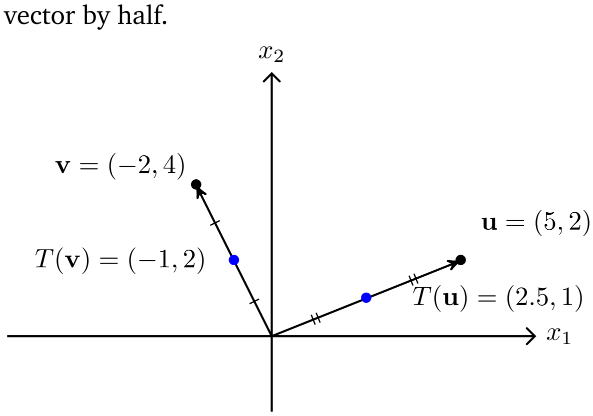
$$\xrightarrow{R_3 = R_3 - 2R_2} \begin{bmatrix} 1 & -4 & 7 & -5 & 1 \\ 0 & 1 & -4 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The last column is not a pivot column, so there exists $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{b}$. In other words, $\mathbf{b}$ is in the range of the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$.

Problem 6. Observe that

$$\begin{bmatrix} .5 & 0 \\ 0 & .5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} .5x_1 \\ .5x_2 \end{bmatrix}$$

This is what the textbook calls **contraction** (see Example 4 in page 71). The linear transformation shrinks the *length* of the vector by half.



Problem 7. The key point is to write both

$$\begin{bmatrix} 5 \\ -3 \end{bmatrix} \text{ and } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

as linear combinations of $\mathbf{e}_1$ and $\mathbf{e}_2$. We have

$$\begin{bmatrix} 5 \\ -3 \end{bmatrix} = 5\mathbf{e}_1 - 3\mathbf{e}_2$$

Therefore

$$T\left(\begin{bmatrix} 5 \\ -3 \end{bmatrix}\right) = T(5\mathbf{e}_1 - 3\mathbf{e}_2) = 5T(\mathbf{e}_1) - 3T(\mathbf{e}_2) = 5\mathbf{y}_1 - 3\mathbf{y}_2 = \begin{bmatrix} 13 \\ 7 \end{bmatrix}$$

Similarly, we get

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - x_2 \\ 5x_1 + 6x_2 \end{bmatrix}$$

Problem 8. We have $T(0) = \mathbf{b}$ and $T(0) + T(0) = 2\mathbf{b}$. If T is linear, $\mathbf{b} = 2\mathbf{b}$, therefore $\mathbf{b} = 0$. Conversely, suppose $\mathbf{b} = 0$. Then $T(\mathbf{x}) = A\mathbf{x}$ is a matrix transformation, and matrix transformations are linear.

Problem 9.

(a) Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

If $T(\mathbf{x}) = A\mathbf{x}$, then

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} \text{ but we know } T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$$

Hence $\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$. Similarly, we can show that $\begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$. To conclude, we obtained

$$A = \begin{bmatrix} 7 & -1 \\ -3 & 6 \end{bmatrix}$$

(b) We first discuss the general situation. Denote by $\mathbf{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ be a vector with 1 at ith entry and 0 everywhere else. Note that for any matrix B, $Be_i = \mathbf{b}_i$ where $\mathbf{b}_i$ is the ith column vector of B. Also, for any vector $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, we have $\mathbf{x} = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n$.

Now back to the situation of the problem, we name $\mathbf{a}_i = T(\mathbf{e}_i)$. We can form the matrix $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$ which we can see is an $m \times n$ matrix. Then

$$T(\mathbf{x}) = x_1T(\mathbf{e}_1) + \cdots + x_nT(\mathbf{e}_n) = x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n = A\mathbf{x}$$

which is the desired result.

Problem 10. Rewrite $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix}$ and $B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix}$. Let $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Then

$$T(\mathbf{e}_1) = \mathbf{a}_1, T(\mathbf{e}_1) = \mathbf{a}_2, S(\mathbf{e}_1) = \mathbf{b}_1, S(\mathbf{e}_2) = \mathbf{b}_2$$

Now $T(S(\mathbf{e}_1)) = T(\mathbf{b}_1) = T(b_{11}\mathbf{e}_1 + b_{21}\mathbf{e}_2) = b_{11}\mathbf{a}_1 + b_{21}\mathbf{a}_2 = \begin{bmatrix} b_{11}a_{11} + b_{21}a_{12} \\ b_{11}a_{21} + b_{21}a_{22} \end{bmatrix}$. We can do the same and obtain

$$T(S(\mathbf{e}_2)) = T(\mathbf{b}_2) = T(b_{12}\mathbf{e}_1 + b_{22}\mathbf{e}_2) = b_{12}\mathbf{a}_1 + b_{22}\mathbf{a}_2 = \begin{bmatrix} b_{12}a_{11} + b_{22}a_{12} \\ b_{12}a_{21} + b_{22}a_{22} \end{bmatrix}$$

which is the what we want.