Worksheet 7	**Name:**	______________________
MATH 240 (Spring 2024)	**Section:**	0111(12PM-1PM) / 0121(1PM-2PM)
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1. **Suggested Problems**

Problem 1 (1.9.2). $T : \mathbb{R}^3 \to \mathbb{R}^2$, $T(\mathbf{e}_1) = (1,3), T(\mathbf{e}_2) = (4,2)$, and $T(\mathbf{e}_3) = (-5,4)$ where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are the columns of the 3×3 identity matrix. Find the standard matrix of T.

Problem 2 (1.9.3). $T : \mathbb{R}^2 \to \mathbb{R}^2$ rotates points (about the origin) through $3\pi/2$ radians (in the counterclockwise direction). Find the standard matrix of T.

Problem 3 (1.9.17). Show that

$$T(x_1, x_2, x_3, x_4) = (0, x_1 + x_2, x_2 + x_3, x_3 + x_4)$$

is a linear transforamtion by finding a matrix that implements the mapping. Note that $x_1, x_2, \ldots$ are not vectors but are entries in vectors.

Problem 4 (1.9.28). **(T/F)** Not every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a matrix transformation.

Problem 5 (1.9.31). **(T/F)** A is a 3×2 matrix, then the transformation $\mathbf{x} \mapsto A\mathbf{x}$ cannot be one-to-one.

Problem 6 (1.9.32). **(T/F)** A is a 3×2 matrix, then the transformation $\mathbf{x} \mapsto A\mathbf{x}$ cannot map $\mathbb{R}^2$ onto $\mathbb{R}^3$.

2. **Additional Problems**

Problem 7. Prove that projection $T : \mathbb{R}^2 \to \mathbb{R}^2$ onto the x_1-axis is neither one-to-one nor onto.

Problem 8. Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation corresponding to i) rotating $\frac{\pi}{2}$ radians counterclock-wise, then ii) reflecting with respect to the x_2-axis, and then iii) horizontally shearing by a factor of 2 units.

(a) What is the standard matrix of T?

(b) What is the linear transformation that *undoes* T? To be precise, find a linear transformation $S : \mathbb{R}^2 \to \mathbb{R}^2$ such that $S(T(\mathbf{x})) = \mathbf{x}$. (**Hint:** if you horizontally sheared by a factor of 2 units, then horizonally shearing by -2 units will *undo* the transformation.)

Let $f : A \to B$ be a function (or a transformation). The function $g : B \to A$ is a left inverse if $g(f(a)) = a$ for all $a \in A$. The function is called a **right inverse** if $f(g(b)) = b$ for all $b \in B$.

Problem 9. Prove that

(a) A function f is one-to-one if and only if f has a left inverse.

(b) A function f is onto if and only if f has a right inverse.

(c) A function g is called an **inverse** of f if g is both a left inverse and a right inverse. Show that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then g is also a linear transformation.

Problem 1. The image $T(\mathbf{e}_i)$ of $\mathbf{e}_i$ gives the ith column of the standard matrix A of T. Therefore

$$A = \begin{bmatrix} 1 & 4 & -5 \\ 3 & 2 & 4 \end{bmatrix}$$

Problem 2. Recall that the matrix that corresponds to rotation about the origin through θ radians in the counterclockwise direction

$$A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Then

$$\cos\left(\frac{3\pi}{2}\right) = 0, \quad \sin\left(\frac{3\pi}{2}\right) = -1$$

Therefore

$$A_{3\pi/2} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Problem 3. This can be solved by trial-and-error.

$$\begin{bmatrix} 0 \\ x_1 + x_2 \\ x_2 + x_3 \\ x_3 + x_4 \end{bmatrix} = \begin{bmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}}_{\text{answer}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Problem 4. **False** because all linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ has an associated standard matrix A such that $T(\mathbf{x}) = A\mathbf{x}$.

Problem 5. Let T be a linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is $T : \mathbb{R}^2 \to \mathbb{R}^3$ as the standard matrix is a 3×2 matrix. The linear transformation $T(x_1, x_2) = (x_1, x_2, 0)$ is one-to-one because

$$T(x_1, x_2) = 0 \Rightarrow (x_1, x_2, 0) = 0 \Rightarrow (x_1, x_2) = 0$$

Therefore the statement is **false**.

Problem 6. We have T maps $\mathbb{R}^2$ onto $\mathbb{R}^3$ $\Leftrightarrow$ for every $\mathbf{b} \in \mathbb{R}^3$, $A\mathbf{x} = \mathbf{b}$ has a solution (where A is the standard matrix of T) $\Leftrightarrow$ every row of A must have a pivot position (by Theorem 4). Since there are more rows to columns, the above cannot be true. The statement is **true**.

Problem 7. Recall that the projection onto the x_1-axis is given by $T(x_1, x_2) = (x_1, 0)$. As we have $T(1, 0) = (1, 0) = T(1, 1)$ with $(1, 0) \neq (1, 1)$, T is not one-to-one. Also, $(1, 1)$ is not in the range of T because images of T always needs to have zero in their x_2-coordinate.

Problem 8.

(a) The linear transformation T can be thought as a composition of three linear transformations. Therefore, the standard matrix of T is a product of three matrices. Namely,

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}}_{\text{shearing by 2}} \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}_{x_2\text{-reflection}} \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{\text{rotation by } \pi/2} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$

Notice how the first linear transformation you apply starts from the right.

(b) We want to *undo* the transformation. We begin by shearing by -2, reflecting with respect to the x_2-axis, and rotating $-\pi/2$ radians. Then the standard matrix of the *undo* linear transformation is

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$$

Problem 9.

(a) Suppose f is one-to-one. Then define $g : B \to A$ as follows. Pick $a_0 \in A$. Then define

$$g(b) = \begin{cases} a_b & \text{if } b = f(a_b) \text{ is in the range of } f \\ a_0 & \text{if } b \text{ is not in the range of } f \end{cases}$$

Then by definition, $g(f(a)) = a$. In other words, g is the left inverse of f.

Suppose f has a left inverse g. Then $f(a_1) = f(a_2) \Rightarrow a_1 = g(f(a_1)) = g(f(a_2)) = a_2$. Therefore, f is one-to-one.

(b) Suppose f is onto. Then define $g : B \to A$ as follows: for all $b \in B$, choose a single a_b such that $f(a_b) = b$. There can be many, but we choose one. Then we can define $g(b) = a_b$. The composition $f(g(b)) = f(a_b) = b$, so g is a right inverse of f.

Suppose f has a right inverse. For every $b \in B$, let $a_b = g(b)$. Then $f(a_b) = f(g(b)) = b$, so f is onto.

Problem 10. Let g be an inverse of f. Let $\mathbf{b}_1, \mathbf{b}_2 \in \mathbb{R}^m$. Since g is a right inverse of f, f is onto. Therefore, there exists $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n$ such that $f(\mathbf{x}_1) = \mathbf{b}_1$ and $f(\mathbf{x}_2) = \mathbf{b}_2$. In fact, $\mathbf{x}_1 = g(\mathbf{b}_1)$ and $\mathbf{x}_2 = g(\mathbf{b}_2)$. ($f(\mathbf{x}_i) = \mathbf{b}_i \Rightarrow g(\mathbf{x}_i) = g(\mathbf{b}_i) \Rightarrow \mathbf{x}_i = g(\mathbf{b}_i)$) Then $g(\mathbf{b}_1 + \mathbf{b}_2) = g(f(\mathbf{x}_1) + f(\mathbf{x}_2)) = g(f(\mathbf{x}_1 + \mathbf{x}_2)) = \mathbf{x}_1 + \mathbf{x}_2 = g(\mathbf{b}_1) + g(\mathbf{b}_2)$. Similarly, let $\mathbf{b} \in \mathbb{R}^m$ and $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{b} = f(\mathbf{x})$. Then $g(\mathbf{b}) = g(f(\mathbf{x})) = \mathbf{x}$.

$$g(c\mathbf{b}) = g(cf(\mathbf{x})) = g(f(c\mathbf{x})) = c\mathbf{x} = cg(\mathbf{b})$$

Therefore, g is linear.