Worksheet 8	**Name:**	____________________
MATH 240 (Spring 2024)	**Section:**	0111(12PM-1PM) / 0121(1PM-2PM)
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Problem 1 (Practice Exam 1.6). Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a transformation where $T(x_1, x_2, x_3) = (x_1x_3, x_2)$. Show T is NOT a linear transformation.

Problem 2 (Practice Exam 1.7). Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a transformation where $T(x_1, x_2, x_3) = (x_1 + x_3, x_2)$. Find the standard matrix representation of the transformation.

Problem 3 (Practice Exam 1.8). Suppose you have 5 vectors in $\mathbb{R}^7$, and none are the zero vector, and every vector is different. You create a matrix where each column is one of those vectors. Treating this matrix as the standard matrix of a linear transformation, is the transformation sometimes, always, or never one-to-one?

Problem 4 (Practice Exam 1.9). Let

$$A = \begin{bmatrix} -1 & 0 \\ -1 & 1 \\ 6 & -3 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 3 & 0 \\ 0 & 4 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 1 \\ 3 & 0 & -1 \\ 0 & 4 & 4 \end{bmatrix}$$

Compute the following, if it exists. If it does not, just write DNE.

(a) CC^T,

(b) $(B + A)^2$,

(c) AB,

(d) CB^T,

(e) A^TC

Problem 5 (Practice Exam 1.10). If the matrix of a linear transformation is given by

$$\begin{bmatrix} 1 & -1 & 4 \\ -2 & 0 & 2 \\ -3 & 4 & -8 \end{bmatrix}$$

Is this transformation one-to-one? Onto? Justify.

Problem 6 (Fall 2015, 1.c) & 1.d)). For each of the following statements, determine TRUE or FALSE. If FALSE, find a counterexample. If TRUE, justify.

(a) If C, D are square matrices of the same size, then $(C + D)(C - D) = C^2 - D^2$,

(b) **(T/F)** If the $n \times n$ matrices A and B each have n pivot positions, then A can be transformed into B by elementary row operations.

Problem 7 (Variant of Fall 2015, 4.a) & 4.b)). Consider the set V of all vectors of the form $\begin{bmatrix} a - 2b + 5c \\ 2a + 5b - 8c \\ -a - 4b + 7c \\ 3a + b + c \end{bmatrix}$ with a, b, c in $\mathbb{R}$ and $\mathbf{u} = \begin{bmatrix} 6 \\ -6 \\ 6 \\ 4 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -2 \\ 5 \\ 4 \\ 1 \end{bmatrix}$.

(a) Justify why the vectors $\mathbf{u}$ and $\mathbf{v}$ are in V.

(b) Are $\mathbf{u}$ and $\mathbf{v}$ linearly independent? Justify your answer.

Problem 8 (Fall 2016, 1). Consider the augmented matrices $\begin{bmatrix} 5 & 2 & 0 & 1 & 7 \\ 0 & 1 & -4 & 4 & 3 \\ 0 & 0 & 2 & 2 & -1 \end{bmatrix}$ and $\begin{bmatrix} 3 & 0 & -2 & 0 \\ 1 & -1 & 4 & 0 \\ 5 & 1 & 2 & 0 \end{bmatrix}$ corresponding to the matrix equations $A\mathbf{x} = \mathbf{b}$ and $C\mathbf{x} = 0$, respectively.

(a) Write the linear system associated to equation $A\mathbf{x} = \mathbf{b}$ and express the solution $\mathbf{x}$ in parametric vector form. Does this equation have a unique solution?

(b) Does the equation $C\mathbf{x} = 0$ have nontrivial solution? Explain. What does this say about the columns of C?

Problem 9 (Fall 2016, 2a)-c)). Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by the formula $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - 3x_2 + 7x_3 \\ 5x_2 - x_3 \\ 2x_3 \end{bmatrix}$.

(a) What is the standard matrix for T?

(b) What is the standard matrix for the transformation $T \circ T$ mapping $\mathbf{x}$ to $T(T(\mathbf{x}))$?

(c) Is T 1-1? Is T onto? Justify.

Problem 1. T is not linear because

$$T(2 \cdot (1,0,1)) = (4,0) \neq (2,0) = 2T(1,0,1)$$

Problem 2. We have $T(1,0,0) = (1,0)$, $T(0,1,0) = (0,1)$, and $T(0,0,1) = (1,0)$, so the standard matrix is

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Problem 3. The linear transformation mentioned in the problem is sometimes one-to-one. Recall that a linear transformation is one-to-one if and only if the columns of its standard matrix is linearly independent. Consider

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The columns of A is linearly independent columns, but the columns of B is not linearly independent as the fourth and the fifth column of B are the same.

Problem 4.

(a)

$$CC^T = \begin{bmatrix} 0 & 1 & 1 \\ 3 & 0 & -1 \\ 0 & 4 & 4 \end{bmatrix} \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 4 \\ 1 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 8 \\ -1 & 10 & -4 \\ 8 & -4 & 32 \end{bmatrix}$$

(b) A and B have differeent size, so cannot add. DNE

(c)

$$AB = \begin{bmatrix} -1 & 0 \\ -1 & 1 \\ 6 & -3 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 3 & 0 \\ 0 & 4 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 0 \\ 1 & 1 & -1 \\ -6 & 6 & 3 \\ 0 & 8 & -2 \end{bmatrix}$$

(d)

$$CB^T = \begin{bmatrix} 0 & 1 & 1 \\ 3 & 0 & -1 \\ 0 & 4 & 4 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 3 & 4 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ -3 & 1 \\ 12 & 12 \end{bmatrix}$$

(e) A^T has size 2×4 and C has size 3×3, so cannot multiply. DNE

Problem 5.

$$\begin{bmatrix} 1 & -1 & 4 \\ -2 & 0 & 2 \\ -3 & 4 & -8 \end{bmatrix} \xrightarrow[R_3 = R_3 + 3R_1]{R_2 = R_2 + 2R_1} \begin{bmatrix} 1 & -1 & 4 \\ 0 & -2 & 10 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow[R_3 = R_3 + \frac{1}{2}R_2]{R_2 = -\frac{1}{2}R_2} \begin{bmatrix} 1 & -1 & 4 \\ 0 & 1 & -5 \\ 0 & 0 & 9 \end{bmatrix}$$

Every row has a pivot, so the associated transformation is onto. To check the columns are linearly independent, let $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the solution to the homogeneous system. Then we use back-substitution. $9x_3 \Rightarrow x_3 = 0$. $x_2 - 5x_3 = 0 \Rightarrow x_2 = 0$. And, similarly, $x_1 - x_2 + 4x_3 = 0 \Rightarrow x_1 = 0$. Therefore, the trivial solution is the ONLY solution. The associated transformation is one-to-one. In particular, the linear transformation is both one-to-one and onto.

Problem 6.

(a) We have

$$(C + D)(C - D) = C^2 - CD + DC - D^2$$

So we need $CD = DC$ for the equality to hold. This is not always true as

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The statement if FALSE.

(b) Note that the *inverse* of an elementary row operation is again an elementary row operation. If C and D are matrices such that $C \to D$ (i.e. C can be transformed to D by elementary row operations), then $D \to C$. By assumption, both $A \to I_n$ and $B \to I_n$ where I_n the $n \times n$ identity matrix. Therefore $A \to I_n \to B$, so it is TRUE.

Problem 7.

(a) $(a, b, c) = (1, 0, 1)$ gives $\mathbf{u}$ and $(a, b, c) = (0, 1, 0)$ gives us $\mathbf{v}$.

(b) Suppose $\mathbf{u} = c\mathbf{v}$. Then by the first entries $6 = -2c \Rightarrow c = -3$. This contradicts the fourth entry which says $c = 4$. Therefore, they are linaerly independent.

Problem 8.

(a) The augmented matrix is already in echelon form, so we write in parametric vector form using back-substitution. Note that we have x_4 is a free variable.

$$\begin{cases} 5x_1 + 2x_2 + x_4 & = 7 \\ x_2 - 4x_3 + 4x_4 & = 3 \\ 2x_3 + 2x_4 & = 1 \end{cases}$$

Then we first get $x_3 = \frac{1}{2} - x_4$. Then $x_2 = 3 + 4x_3 - 4x_4 = 3 + 2 - 4x_4 - 4x_4 = 5 - 8x_4$. Finally, $5x_1 = 7 - 2x_2 - x_4 = 7 - 10 + 16x_4 - x_4 = -3 + 15x_4$. Therefore

$$\mathbf{x} = \begin{bmatrix} -\frac{3}{5} \\ 5 \\ \frac{1}{2} \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ -8 \\ -1 \\ 1 \end{bmatrix}$$

(b) For homogenous system, it is enough to look at the coefficient matrix C

$$\begin{bmatrix} 3 & 0 & -2 \\ 1 & -1 & 4 \\ 5 & 1 & 2 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & -1 & 4 \\ 3 & 0 & -2 \\ 5 & 1 & 2 \end{bmatrix} \xrightarrow[R_3 = R_3 - 5R_1]{R_2 = R_2 - 3R_1} \begin{bmatrix} 1 & -1 & 4 \\ 0 & 3 & -14 \\ 0 & 6 & -18 \end{bmatrix}$$

$$\xrightarrow{R_3 = R_3 - 2R_2} \begin{bmatrix} 1 & -1 & 4 \\ 0 & 3 & -14 \\ 0 & 0 & 10 \end{bmatrix}$$

Using back-substitution (see Problem 5), to see that the only solution to the homogeneous system is the trivial solution. This says that the columns are linearly independent.

Problem 9.

(a)

$$\begin{bmatrix} 2 & -3 & 7 \\ 0 & 5 & -3 \\ 0 & 0 & 2 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 2 & -3 & 7 \\ 0 & 5 & -3 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 & 7 \\ 0 & 5 & -3 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & -21 & 37 \\ 0 & 25 & -21 \\ 0 & 0 & 4 \end{bmatrix}$$

(c) The columns of the standard matrix is linearly independent (back-substitution) and every row has a pivot position. Therefore, T is both 1-1 and onto.