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| **Worksheet 9** | **Name:** | ________________________ |
| **MATH 240 (Spring 2024)** | **Section:** | 0111(12PM-1PM) / 0121(1PM-2PM) |
| **Feb 27, 2024** | **TA:** | Shin Eui Song |

**Problem 1** (Spring 2018, Problem 2c)**.** Consider the parameters $a$ and $b$ and the system

$$\begin{cases} -2x_1 + ax_2 = 1 \\ 6x_1 + bx_2 = -2 \end{cases}$$

(a) i) Write its augmented matrix. ii) Write the system as a vector equation. iii) Write the system as a matrix equation.

(b) Is this system homogeneous?

(c) Find some values of the parametrs $a$ and $b$ for which the system is inconsistent.

**Problem 2** (Practice Exam 1.1)**.** Solve the following systems by row reducing the augmented matrix. If there is a unique solution, state what it is. If there is no solution, explain why. If there are infinite solutions, express your solution in parametric vector form.

(a)

$$\begin{aligned} x + \ y + \ z &= 3 \\ y + 2z &= -5 \\ x + 2y + 4z &= -4 \end{aligned}$$

(b)

$$\begin{aligned} x_1 - 3x_2 + 2x_3 - 5x_4 &= 3 \\ 2x_1 - 6x_2 + x_3 - 7x_4 &= 2 \\ x_1 - 3x_2 - 4x_3 + x_4 &= -5 \end{aligned}$$

**Problem 3** (Practice Exam 1.2 and more)**.** For the following UNRELATED statements, determine if the statement is true or false. If it is TRUE, simply state TRUE. If it is FALSE, **provide an explicit counterexample** i.e. an explicit example that shows the statement is false.

(a) Suppose $A$ is a $3 \times 4$ matrix that is the standard matrix of a linear transformation. Then the transformation is always onto.

(b) Suppose $A$ is a $3 \times 4$ matrix that is the standard matrix of a linear transformation. Then the transformation is never one-to-one.

(c) Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be vectors in $\mathbb{R}^n$. If $n \geq 3$, then the set of vectors must be linearly independent.

(d) Let $A$ be $m \times n$ matrix. If $m > n$, then the set of ROW vectors (each row represents 1 vector) must be linearly dependent.

(e) If $A$ is the matrix representation of a linear transformation and we know the columns of $A$ form a linearly independent set, then the transformation is always onto.

(f) If the coefficient matrix $A$ has a pivot position in every column, then equation $A\mathbf{x} = \mathbf{b}$ is consistent.

**Problem 4** (Practice Exam 1.3)**.** Determine if $(0, 10, 8)$ lies in

$$\mathrm{Span}(\{(-1, 2, 3), (1, 3, 1), (1, 8, 5)\}).$$

If it does lie in the span, find an explicit linear combination.

Is the set $\mathrm{Span}(\{(-1, 2, 3), (1, 3, 1), (1, 8, 5)\})$ linearly independent?
What about the set $\{(-1, 2, 3), (1, 3, 1), (1, 8, 5)\}$?

**Problem 5** (Practice Exam 1.4)**.** Suppose $\mathbf{v}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$. If we add another vector $\mathbf{v}_{m+1}$, will $\mathbf{v}$ sometimes, always, or never be in $\mathrm{Span}(\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m, \mathbf{v}_{m+1}\})$?

**Problem 6.** Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that $T(1, 2) = (5, 4, 3)$ and $T(4, 3) = (1, 1, 2)$. Find $T(2, -1)$.

**Problem 1.**

(a) i)

$$\begin{bmatrix} -2 & a & 1 \\ 6 & b & -2 \end{bmatrix}$$

ii)

$$x_1 \begin{bmatrix} -2 \\ 6 \end{bmatrix} + x_2 \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

iii)

$$\begin{bmatrix} -2 & a \\ 6 & b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

(b) The system is not homogeneous because the right-hand side is not zeroes.

(c)

$$\begin{bmatrix} -2 & a & 1 \\ 6 & b & -2 \end{bmatrix} \xrightarrow{R_2 = R_2 + 3R1} \begin{bmatrix} -2 & a & 1 \\ 0 & b+3a & 1 \end{bmatrix}$$

Then it would be inconsistent when $b + 3a = 0$. Hence $a = -1$ and $b = 3$ works.

**Problem 2.**

(a)

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & -5 \\ 1 & 2 & 4 & -4 \end{bmatrix} \xrightarrow{R_3 = R_3 - R_1} \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & -5 \\ 0 & 1 & 3 & -7 \end{bmatrix} \xrightarrow{R_3 = R_3 - R_2} \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

Then

$$\begin{cases} x_3 &= -2 \\ x_2 &= -5 - 2x_3 = -1 \\ x_1 &= 3 - x_2 - x_3 = 6 \end{cases}$$

(b)

$$\begin{bmatrix} 1 & -3 & 2 & -5 & 3 \\ 2 & -6 & 1 & -7 & 2 \\ 1 & -3 & -4 & 1 & -5 \end{bmatrix} \xrightarrow[R_3 = R_3 - R_1]{R_2 = R_2 - 2R_1} \begin{bmatrix} 1 & -3 & 2 & -5 & 3 \\ 0 & 0 & -3 & 3 & -4 \\ 0 & 0 & -6 & 6 & -8 \end{bmatrix}$$

$$\xrightarrow[R_3 = R_3 - 2R_2]{R_2 = -\frac{1}{3}R_2} \begin{bmatrix} 1 & -3 & 2 & -5 & 3 \\ 0 & 0 & 1 & -1 & \frac{4}{3} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then

$$\mathbf{x} = \begin{bmatrix} 3 + 3x_2 - 2x_3 + 5x_4 \\ x_2 \\ \frac{4}{3} + x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} + 3x_2 + 3x_4 \\ x_2 \\ \frac{4}{3} + x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ 0 \\ \frac{4}{3} \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

**Problem 3.**

(a) FALSE, consider the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The span of the column vectors is clearly $\{0\}$ which is not $\mathbb{R}^3$. The associated linear transformation is not onto.

(b) TRUE, there are four vectors with three entries. Therefore, the set of column vectors must be linearly dependent.

(c) FALSE, for $n = 3$, the set

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

is linearly dependent.

(d) TRUE, for ROW vectors, $m$ is the number of vectors and $n$ is the number of entries. Since $m > n$, the set of row vectors is linearly dependent.

(e) FALSE, consider

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Then $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is not in the span of column vectors, so the associated transformation is not onto.

(f) FALSE, consider the coefficient matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

This matrix has as pivot position in every column, but the quation $A\mathbf{x} = \begin{bmatrix} 5 & 3 & 1 \end{bmatrix}$ does not have a solution.

**Problem 4.** We may solve linear system associated to the augmented matrix

$$\begin{bmatrix} -1 & 1 & 1 & 0 \\ 2 & 3 & 8 & 10 \\ 3 & 1 & 5 & 8 \end{bmatrix} \xrightarrow[R_3 = R_3 + 3R1]{R_2 = R_2 + 2R_1} \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & 5 & 10 & 10 \\ 0 & 4 & 8 & 8 \end{bmatrix} \xrightarrow[R_3 = \frac{1}{4}R_3]{R_2 = \frac{1}{5}R_2} \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 2 \end{bmatrix}$$

$$\xrightarrow{R_3 = R_3 - R_2} \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

There are infinitely many solutions with free variable $x_3$. Let $x_3 = 0$, then $x_2 = 2$ and $x_1 = 2$. In other words,

$$\begin{bmatrix} 0 \\ 10 \\ 8 \end{bmatrix} = 2 \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 8 \\ 5 \end{bmatrix}$$

The both sets $\mathrm{Span}(\{(-1,2,3),(1,3,1),(1,8,5)\})$ and $\{(-1,2,3),(1,3,1),(1,8,5)\}$ are linearly dependent. To see this, consider the augmented matrix for the homogeneous system. Repeating the same row reduction,

$$\begin{bmatrix} -1 & 1 & 1 & 0 \\ 2 & 3 & 8 & 0 \\ 3 & 1 & 5 & 0 \end{bmatrix} \to \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

If $x_3 = 1$, then $x_2 = -2$. So $x_1 = -1$. Therefore,

$$\begin{bmatrix} 1 \\ 8 \\ 5 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

This shows that both sets are linearly dependent as

$$\{(-1,2,3),(1,3,1),(1,8,5)\} \subset \mathrm{Span}(\{(-1,2,3),(1,3,1),(1,8,5)\}).$$

**Problem 5.** $\mathbf{v}$ is ALWAYS in $\mathrm{Span}(\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m, \mathbf{v}_{m+1}\})$. By assumption,

$$\begin{aligned} \mathbf{v} &= a_1\mathbf{v}_1 + \cdots + a_m\mathbf{v}_m \\ &= a_1\mathbf{v}_1 + \cdots + a_m\mathbf{v}_m + 0\mathbf{v}_{m+1} \in \mathrm{Span}(\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m, \mathbf{v}_{m+1}\}) \end{aligned}$$

**Problem 6.** First, we need to write $(2,-1)$ as a linear combination of $(1,2)$ and $(4,3)$. To do this, one can solve the augmented matrix

$$\begin{bmatrix} 1 & 4 & 2 \\ 2 & 3 & -1 \end{bmatrix}$$

Or simply observe that

$$(4,3) - 2(1,2) = (2,-1)$$

Then

$$T(2,-1) = T((4,3) - 2(1,2)) = T(4,3) - 2T(1,2) = (1,1,2) - 2(5,4,3) = (-9,-7,-4).$$