# Introduction to Proof in Analysis - 2020 Edition

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(with contributions from Elizabeth Hughes)



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# Chapter 1

# Introduction

## 1.1 Purpose

Mathematics explores a universe inspired by, but different from, the real world we live in. It is full of wonderfully beautiful phenomena, but whose truth can only be validated by rigorous logical arguments, which we call **proofs**. The *purpose* of this course is to introduce you to this universe, to help you learn and apply the language and techniques of mathematical proof, and in the process to prepare you for Math 410.

Becoming familiar with a new language can be a frustrating process, especially when it is not simply a matter of translation of English into a different language. There are two fundamental differences:

First, Mathematical language is how we describe and validate mathematical phenomena (different from what we use normal language for) and so *we have to learn to understand the phenomena we are describing.*

Second, in normal spoken language we often speak "approximately" and our audience can easily figure out what we mean from the context. *In Mathematics it is essental to use the language absolutely precisely, because it is the unique tool we have to validate the truth of our assertions.*

As you come to grips with this course I would *strongly* encourage you to take full advantage of my availability to support your efforts outside this class.

Here are some of the ways available to you. You can do this individually or in groups!

- Ask questions in class!!
- Ask questions before or after class!!
- Come to office hours!!
- Make appointments to see me outside office hours!!
- Email me with questions!!

Now this course may seem quite different from your previous math courses, where a major focus may have been on how to construct derivatives and integrals, to multiply and invert matrices, and to solve linear equations.

I would like to illustrate that difference with an example of a proof of a theoretical mathematical result, which still only uses facts you may have seen in your calculus classes. For this example we recall that [c,d] denotes the closed interval of real numbers x satisfying $c \leq x \leq d$. Similarly, [c,a), (a,b) and b,d] denote respectively the intervals of real numbers x satisfying $c \leq x < a, a < x < b$ and $b < x \leq d$.

**Example:** Recall that a *real polynomial of degree n* is a real-valued function of the form

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n,$$

in which the $a_k$ are real constants and $a_n \neq 0$. A *real zero* of such a polynomial is a real number $b$ such that $f(b) = 0$.

**Theorem:** Suppose such a real polynomial $f(x)$ of degree n and with $a_n = 1$ has n distinct real zeros,

$$b_1 < ... < b_n.$$

Let $[c, d]$ be an interval such that $c < b_1 < ... < b_n < d$,

Then there is a positive number $\varepsilon > 0$ such that if $g(x)$ is any differentiable function for which

$$|g(x)| < \varepsilon, \quad \text{for every} \quad x \in [c, d],$$

then $f(x) + g(x)$ has at least n zeros in $[c, d]$.

**Proof:** We do this in Steps

**Step One:** $f(x) = (x - b_1)(x - b_2)...(x - b_n)$.

You may already have seen this, but in any case I am going to prove it a little later, and so will skip the proof for now.

**Step Two:** In each of the intervals $[c, b_1), (b_1, b_2), ..., (b_n, d]$, $f(x)$is never zero and therefore has constant sign.

*proof:* Since $f(x) = (x - b_1)(x - b_2)...(x - b_n)$, it follows that $f(x)$ is non-zero in each of these intervals. Moreover, if there were two points $x < y$ in one of these intervals at which $f$ had opposite signs, then by the
Intermediate Value Theorem, for some $z$ between $x$ and $y$, it would be true that $f(z)$ was zero, which is not the case.

**Step Three:** The values $f'(b_k)$ of the derivative of $f$ are non-zero.

*proof:* Write

$$f(x) = (x - b_k)h(x).$$

By the product rule,

$$f'(b_k) = h(b_k) \neq 0.$$

**Step Four:** *The signs of the values of $f(x)$ in the intervals $(c, b_1), (b_1, b_2), ..., (b_n, d)$ alternate from interval to interval.*

*proof:* First consider the case that $f(x)$ is positive in the interval to the right of $b_k$. Then we compute the derivative at $b_k$ by taking the limit as $x$ approaches $b_k$ from the right:

$$f'(b_k) = lim_{h \to 0} \frac{f(b_k + h) - f(b_k)}{h} \geq 0.$$

Since $f'(b_k) \neq 0, f'(b_k) > 0$.

Since the derivative at $b_k$ is positive the function must be increasing in an interval around $b_k$. Since its value at $b_k$ is zero, it must be negative in the interval to the left of $b_k$.

The other case is when $f(x)$ is negative in the interval to the right of $b_k$. Then, when we compute the derivative as a limit from the right we see that $f'(b_k) \leq 0$ and so it must be negative. In this case the function is decreasing near $b_k$, and so it must be positive to the left.

**Step Five:** *Completion of the proof:*

Pick any numbers:

$$y_0 \in (c, b_1)$$
$$y_i \in (b_i, b_{i+1}), \quad 1 \leq i \leq n - 1, \quad \text{and}$$
$$y_n \in (b_n, d).$$

Then each $f(y_i) \neq 0$. Choose $\varepsilon$ to be a positive number such that

$$\varepsilon < |f(y_i)|, \quad i = 0, \cdots, n.$$

Now suppose $g(x)$ is any differentiable function in the interval $[c, d]$ such that

$$|g(x)| < \varepsilon, \quad \text{for every} \quad x \in [c, d].$$

Then for each $i$, $f(y_i)+g(y_i)$ has the same sign as $f(y_i)$. Therefore, $f(y_i)+g(y_i)$ and $f(y_{i+1})+g(y_{i+1})$ have opposite signs.

Therefore, by the Intermediate Value Theorem, there are points $z_i$ in the intervals $(y_i, y_{i+1})$ such that

$$(f+g)(z_i) = 0.$$

q.e.d.

## 1.2 Expectations:

*What you can expect from me:*

1. A clear statement of what material will be covered, of what you will be expected to learn to do, and of how that will be assessed:

   (a) The material to be covered is explicitly stated in the Syllabus.

   (b) You will be expected to learn to solve problems and express the solutions as formal, clearly written mathematical proofs.

   You will *not* be asked to repeat proofs of theorems and definitions.

   However, *unless you know these cold* you will *not* be able to produce correctly written solutions.

   (c) Assessment will be through weekly homework assignments, 3 term tests, and a final exam. Your work will be graded on how well you meet the following requirements:

   Your answers must be a sequence of statements, each following logically from the previous ones, with an explanation as to why. They must be
   – clear, precise, and unambiguous,
   – written in complete, grammatical sentences, and
   – for homeworks only, typed or written legibly in ink.

   They may rely **only** on:
   – any statement already proved in class;
   – any statement proved in an earlier part of the text;
   – any statement in any earlier exercise.

2. *Good access to me outside lectures*

   (a) Office hours are posted in the Syllabus.

(b) I am often in my office outside office hours, and you are always welcome to drop in and see if I am free to answer a question or explain something.

(c) You can always make an appointment by email to see me.

(d) I regularly check my email and respond to questions and requests (but not too late at night!).

3. Readiness to respond to questions, and to clarify material you have not understood.

   (a) It is almost a mathematical theorem that working with me outside class can be a big factor in your success.

   (b) My basic philosophy is: *There are no stupid questions!*

   (c) I will always try to help you when you are stuck, but I can be more useful to you if you have first tried to unstick yourself.

*What you should expect from yourselves:*

1. Stay engaged with the course.

2. Be proactive: always ask a question if you do not understand something, or do not see how to attack a problem.

3. Join a study group if that is helpful, but never as a freeloader.

4. Be prepared to work at finding a solution to a problem: problems are not just variations on examples from class.

5. **Master and be comfortable with** the meaning of the words and symbols as they are introduced.

6. Stay current with the material, and do the assigned readings **as assigned**.

7. Do the assigned homework each week.

**Students who invest this time and effort usually do well. Students who do not make this effort usually do not.**

# Chapter 2

# Mathematical Proof

## 2.1 The Language of Mathematics

You learn a second language most easily by speaking it with others to whom it comes naturally. You learn to drive a car by driving it and to walk by walking. You learn to write/speak Mathematics by writing it and presenting it and getting feedback when you get it right and how to correct it when you don't. The golden rule when writing: **never write anything whose meaning is unclear to yourself!** You can also use this text to find many detailed examples of how to write a proof correctly.

Mathematical statements may be definitions, or logical statements, and can express a complicated idea in a few words or symbols, as the following examples show. **Thus until one gets used to the language it really can take a mental effort to understand a mathematical statement.**

Now to get started, here are the some principal aspects to Mathematical language:

It relies entirely on its *precision*, which means that each word and symbol is exactly *defined* and can only be used that way. These words **name** mathematical objects, including the natural numbers, the integers, the rationals, the real numbers, sets, maps, functions and many other things. You will need to **know and understand these objects well**, and so you will need to be comfortable with their names.

Mathematical symbols are used to convert long sentences into short statements and so to read and write Mathematics you will also have to be comfortable with the meaning of the symbols.

Mathematics also has some of its own particular expressions:

1. **q.e.d.** stands for quid erat demonstrandum, which is Latin for "what was to be proved". It is used at the end of a proof to signal that the proof is complete.

2. **Let....:** This is a statement which **defines** some terminology or **establishes notation**. As an example: Let "'n"' be a natural number greater than 5.

3. **Theorem, proposition, lemma:** These are names for mathematical statements that are going to be proved.

To summarize: it is essential that you ***internalize*** the definitions and notation rather than simply memorize them. Indeed, when learning to speak or write a new language you need to be able to use the words spontaneously without having to call up each corresponding English word and then translate it. **In the same way, you need to be able to speak/write mathematics, not just remember the dictionary of definitions.** This is essential, because

**When you write a statement in Mathematics it must say to any reader exactly what you mean**

Finally, Mathematics uses a number of logical expressions for you to become familiar with, since they are how the deductions in a proof are expressed. Everyone, even at an early age, learns what it means for a statement to follow logically from others, and sometimes it will be obvious that the statement is correct. So, writing proofs and solving problems does not require you to **learn** to think logically, since you already know how to do that. What it does require is for you to learn how to use that ability in the world of Mathematics.

## 2.2 What is a Proof in Mathematics?

A **proof** in Mathematics is a sequence of statements which establish that certain assumptions (**the hypotheses**) imply that a certain statement **(the conclusion)** is true. This sequence of statements must satisfy:

1. Each is clear and unambiguous,

2. Each is true, and its truth follows immediately from the truth of the preceding statements and the hypotheses.

3. The final statement is the conclusion.

**Example 1** *1. Prove that if a student is in Math 310 (the hypothesis), s/he is registered at the University of Maryland(the conclusion).*

***Proof:*** *To be in Math 310 you must be registered at the University. Therefore, since the student is in Math 310, s/he is registered at the University. q.e.d.*

2. *Prove that if $x$ is a real number and $x > 1$ then $x^2 > x$.*

   ***Proof:*** $x^2 - x = x(x-1)$.
   *Since $x > 1$, $x$ and $x - 1$ are both positive.*
   *Therefore $x(x-1)$ is positive.*
   *Therefore $x^2 > x$. q.e.d.*

3. *If $n$ is a natural number and $n > 1$, prove that $n^3 - 1$ is divisible by $n - 1$.*

   ***Proof:*** *let $n$ be a natural number and suppose $n > 1$. Direct multiplication gives $(n-1)(n^2+n+1) = n^3 - 1$.*
   *Since $n^2 + n + 1$ is a natural number, $n^3 - 1$ is divisible by $n - 1$.*
   *q.e.d.*

## 2.3 To solve a 310 problem

Here are the basic steps

1. **Fully understand** the problem, and the mathematical terms used. In particular, make sure you understand
   (a) what you are given (the **hypotheses**);
   (b) what you are asked to prove (the **conclusion**);
   (c) the **terminology** and the **notation**.
2. **Figure out why** the statement to prove is true.
3. Express your solution in a **formal mathematical proof**, as follows:

   (a) Start your solution with **"We have to prove that ..."** This is what is to be proved.
   (b) Then, beginning with the word "**proof**" provide a sequence of **sentences**, each following logically from the preceding ones and the hypotheses.
   (c) Write so that each sentence has a **single** meaning which will be clear to any reader.
   (d) The final statement should be the **conclusion**, which your proof has now established as true.
   (e) **Note:** When writing mathematics it is almost always useful to label the objects under consideration. This is called "**establishing notation**".

**Tips for Writing Proofs** from Elizabeth (a former Math 310 student):

1. Learning to write proofs is not like learning how to multiply matrices, and it can take mental effort and time.

2. Remain calm; if you put enough time into this class, you will figure out how to write proofs.

3. Give yourself plenty of time to work on a proof.

4. Before you start the proof, look back at the given definitions and lemmas that you think might be useful for this proof. Make sure you fully understand the definitions and lemmas and see if you can find a way to connect them back to what you are trying to prove.

5. Restate definitions in your proofs, so you are less likely to use a definition incorrectly.

6. Working in groups is fine, but make sure you try each exercise thoroughly on your own first. You will not have your group to help when it comes time to take the test.

7. Never hesitate to ask Professor Halperin a question when you are stuck: either in class, in his office, or by email. He will welcome your interest: I know!

8. Most importantly: This class is designed to help you learn how to write proofs. While it is important you understand the concepts, you must be able to use what you learn to complete a correctly written proof in order to succeed.

## 2.4 Sets, numbers, and sequences

**Definition 1** *Sets*

*A* ***set*** *is a specified* ***collection*** *of distinct objects, abstract or concrete, called its* ***elements****.* ***Thus, to define a set, you must specify precisely what are its elements!***

*An element $x$ in a set $S$ is said to* ***belong*** *to $S$, and we denote this by* $\mathbf{x} \in \mathbf{S}$. *We often write*

$$S = \{x, y, z, \cdots\},$$

*where $x, y, z \cdots$ are the elements of $S$.*

**Example 2**

1. *The collection of cars that were parked on campus on July 1, 2000 is a set.*

2. *The collection of students currently enrolled as math majors at Maryland is a set.*

3. *The collection of real polynomials of degree $n$ is a set.*

4. *The collection of $3 \times 3$ matrices with real entries is a set.*

5. *The collection of non-zero complex numbers is a set.*

**Definition 2** ***Numbers***

1. ***N*** *will denote the set of natural numbers, and we write* $\boldsymbol{N} = \{1, 2, 3, \ldots\}$. *Thus* 0 *is not a natural number.*

2. ***Z*** *will denote the set of integers:* $\boldsymbol{Z} = \{0, \pm 1, \pm 2, \pm 3 \ldots\}$.

3. ***Q*** *will denote the set of rational numbers: these are the real numbers which can be written* $p/q$ *with* $p \in \boldsymbol{Z}$ *and* $q \in \boldsymbol{N}$. *Note that* $p/q = r/s$ *if and only if* $ps = rq$.

4. ***R*** *will denote the set of real numbers.*

5. ***C*** *will denote the set of complex numbers.*

6. *An integer* $n$ *is* ***even*** *if it is divisible by* 2*; ii is* ***odd*** *if it is not even.*

7. *A* ***prime number*** *is a natural number* $n$ *which is larger than* 1 *and is not divisible by any natural number except* 1 *and* $n$.

8. *If* $n \in \boldsymbol{N}$ *then* $n$ *is positive and* $-n$ *is negative. If* $a = p/q \in \boldsymbol{Q}$ *and* $p \in \boldsymbol{N}$ *then* $a$ *is positive and* $-a$ *is negative.*

9. *A real number* $x$ ***is larger*** *than a real number* $y$ *if* $x - y$ *is positive.* ***(CAUTION: Note that*** $-1$ ***is larger than*** $-2$***!)***

10. *If* $x$ *is a real number and* $n$ *is a natural number then* $x^n$ *is the real number obtained by multiplying* $x$ *with itself* $n$ *times.*

11. *The* ***absolute value*** *of a real number* $x$ *is denoted by* $|x|$ *and is defined by:*

$$|x| = \begin{cases} x, & x \geq 0, \\ -x, & x < 0 \end{cases}$$

*Note that* $|x| \geq 0$ *for* ***all*** $x \in \boldsymbol{R}$.

12. *A real number,* $x$*, is* ***positive*** *if* $x > 0$ *and* ***negative*** *if* $x < 0$. *Thus* 0 *is neither positive or negative.*

For Exercise 1 below you may find it helpful to read Sec. 2.6 in the text.

**Exercise 1** 1. *Show that for each rational number* $a \in \boldsymbol{Q}$ *exactly one of the following three possibilities is true:*

(a) $a > 0$

(b) $a = 0$

(c) $a < 0$

2. *If* $a, b, c$ *are any three rational numbers, show that*

(a) *If* $a < b$ *and* $b < c$ *then* $a < c$.

(b) *$a + b < a + c$ if and only if $b < c$.*

(c) *Multiplication by a positive preserves inequalities.*

(d) *Multiplication by a negative reverses inequalities.*

3. *If $a \in \boldsymbol{Q}$ then $a > 0$ if and only if $a > 1/n$ for some $n \in \boldsymbol{N}$.*

**Proposition 1** ***The Division Proposition for Numbers:***

*If $p < n$ are natural numbers then for some natural number $m \leq n$ and for some integer $r$ with $0 \leq r < p$, n=mp +r.*

(The number $r$ is called the **remainder**.)

**Proof:** Let $p < n$ be natural numbers. If $k > n$ then for any $r \geq 0$

$$kp \geq k > n.$$

Thus if $mp + r = n$ it must be true that $mp \leq n$ and so $m \leq n$.
Since $1 \leq n$ we can choose $m$ to be the largest natural number such that $mp \leq n$.
Then

$$n - mp \geq 0.$$

On the other hand, if $n - mp \geq p$ then $(n - mp) - p > 0$ and so

$$n - (m + 1)p = (n - mp) - p > 0.$$

Thus $m$ was not the largest natural number which multiplied by $p$ gave an answer at most $n$, and we assumed it was.
Thus $n - mp < p$ and we may set $r = n - mp$. q.e.d.

**Definition 3** *A* ***sequence*** *starting at $k \in \boldsymbol{Z}$ is a* ***list*** *$(x_n)_{n \geq k}$ of objects, possibly with repetitions, and indexed by the integers $n, n \geq k$. If $(x_n)_{n \geq k}$ is a sequence, then $x_n$ is the $n^{th}$* ***term*** *in the sequence.*

**Note:** The objects listed in a sequence form a set, as seen in the first two of the following examples.

**Example 3**

1. *The list $0, 1, 0, , 1, 0, 1, \cdots$ is a sequence. The corresponding set is the collection $\{0, 1\}$ which only has two elements.*

2. *The list $2, 4, 2, 4, 6, 2, 4, 6, 8, 2, 4, 6, 8, 10, \cdots$ is a sequence. The corresponding set is the collection of even natural numbers.*

3. *A sequence $(x_n)_{n \geq 1}$ of numbers is defined by*

$$x_n = 2^n + n^2.$$

4. *A sequence of fractions* $(y_n)_{n\geq 2}$ *is defined by*

$$y_n = \frac{n}{n-1}.$$

**Exercise 2** 1. *Suppose in a room with people and animals, some of the animals may be pets of some of the people. Then for each pair of the following statements decide if one implies the other, and justify your answer.*

   (a) *Every pet in the room has an owner in the room.*
   (b) *Some pet in the room has an owner in the room.*
   (c) *Each person in the room owns a pet in the room.*
   (d) *Some person in the room owns a pet in the room.*

2. *In the previous problem construct for each statement an example for which the statement is false.*

3. *Which of the following are sets? Justify your answers.*

   (a) *Some apples in a grocery store.*
   (b) *Chalk in one math classroom.*
   (c) *The positive rational numbers.*
   (d) *The faculty and staff pictures in the entry lobby of the Math building.*

4. *For each statement below decide if it is true or false and either provide a proof that it is true or give an example to show that it is false.*

   (a) *If* $x \in \boldsymbol{Z}$ *and* $x \geq 0$, *then* $x \in \boldsymbol{N}$.

   (b) *If* $p, q \in \boldsymbol{Z}$ *and* $p + q \in \boldsymbol{N}$, *then* $p \in \boldsymbol{N}$.

   (c) *If* $x \in \boldsymbol{N}, y \in \boldsymbol{Z}$, *and* $z \in \boldsymbol{C}$, *then* $x, y, z \in \boldsymbol{R}$.

   (d) *If* $p, q \in \boldsymbol{N}$, *and* $p \geq 2$ *and* $q \geq 6$, *then* $p + q$ *is even.*

5. *Describe what is wrong with each of the following statements:*

   (a) *Laptops are popular among college students. Therefore at least one student at Maryland has a laptop.*

   (b) *It is true that* $u = 6$ *for every* $x, y \in \boldsymbol{Q}$ *such that* $x + y < u$.

6. *Does (a) or (b) correctly add "such that" to the statement: "every student has a height h"?*

(a) *Every student has a height $h$ such that $h \leq a$ where $a$ is the height of the tallest student.*

(b) *Every student has a height $h$ such that every student is in a class.*

## 2.5 Sums, Products, and the Sigma and Pi Notation

1. The **"Sigma" notation** is used for sums:
   If $v_i$ are numbers or vectors (or anything else we know how to add) then

   $$\sum_{i=1}^{n} v_i = v_1 + v_2 + \cdots v_n.$$

   More generally,

   $$\sum_{i=1}^{n} v_{k_i} = v_{k_1} + v_{k_2} + v_{k_3} + \cdots v_{k_n}.$$

   In this notation the "$i$" just indexes the terms being added or multiplied, and we could use any letter instead without changing the meaning. Thus

   $$\sum_{i=1}^{n} v_i = \sum_{l=1}^{n} v_l = \sum_{q=1}^{n} v_q = v_1 + v_2 + \cdots v_n.$$

   In other words, the "$i$" is a dummy variable just as in calculus, where we have

   $$\int f(x)dx = \int f(u)du$$

2. The **"Pi" notation** is used for products:
   If $v_i$ are numbers then

   $$\prod_{i=1}^{n} v_i = v_1 \cdot v_2 \cdots v_n.$$

   More generally,

   $$\prod_{i=1}^{n} v_{k_i} = v_{k_1} \cdot v_{k_2} \cdot v_{k_3} \cdots v_{k_n}.$$

3. In particular, **if $C$ is a constant**, then

   $$\sum_{i=1}^{n} C = C + \cdots + C \text{ ($n$ times)} = nC$$

   and

   $$\prod_{i=1}^{n} C = C \cdots C = C^n.$$

**Example 4** *Prove that if every student in Math 310 gets at least* $7/10$ *on a homework then the class average is at least* $7/10$*.*

***Proof:*** *Let the number of students in the class be* $n$ *(**Establishes notation.**). Suppose every student has a grade at least* $7/10$*. Let* $x_i$ *be the grade of the* $i^{th}$ *student. Then, by definition, the class average is*

$$\frac{\sum_{i=1}^{n} x_i}{n}.$$

*Since each* $x_i \geq 7$*, therefore*

$$\frac{\sum_{i=1}^{n} x_i}{n} - 7 = \frac{\sum_{i=1}^{n} x_i}{n} - \frac{7n}{n} = \frac{\sum_{i=1}^{n}(x_i - 7)}{n} \geq 0.$$

*Therefore the class average is at least* $7/10$*.*
*q.e.d.*

**Theorem 1** ***(Difference Theorem)*** *For any non-zero real numbers* $a, b \in \boldsymbol{R}$ *and any* $n \in \boldsymbol{N}$*,*

$$b^n - a^n = (b-a)\sum_{i=0}^{n-1} b^i a^{n-1-i}.$$

**Proof:** Note that

$$b(\sum_{i=0}^{n-1} b^i a^{n-1-i}) = \sum_{i=0}^{n-1} b^{i+1} a^{n-1-i} = \sum_{j=0}^{n-1} b^{j+1} a^{n-1-j}.$$

In the second expression set $j+1 = i$.Then

$$\sum_{j=0}^{n-1} b^{j+1} a^{n-1-j} = \sum_{i=1}^{n} b^i a^{n-i},$$

and so

$$b(\sum_{i=0}^{n-1} b^i a^{n-1-i}) = \sum_{i=1}^{n} b^i a^{n-i} = b^n + \sum_{i=1}^{n-1} b^i a^{n-i}.$$

On the other hand,

$$(-a)(\sum_{i=0}^{n-1} b^i a^{n-1-i}) = -\sum_{i=0}^{n-1} b^i a^{n-i} = -a^n - \sum_{i=1}^{n-1} b^i a^{n-i}.$$

Adding these two lines gives the equation of the Theorem.
q.e.d.

## 2.6 Logical Expressions for Proofs

In Mathematics, you will frequently encounter statements about statements! Moreover, just as we use symbols to represent unspecified numbers, functions, and matrices, so also we may use symbols to represent statements!

In Mathematics, we are often asked to decide **which**: is the statement true or is it false, and then to supply a proof. So here are some useful statements connecting the possible truth of two statements, A and B, and some terminology:

1. **A if B.** This means that *A is true if B is true* and may also be phrased as: *B implies A*.

2. **If A then B .** This is the same as *B if A* and so it means thst *if A is true then B is true* and may also be phrased as: *A implies B*. It is denoted by

$$A \Rightarrow B.$$

3. **A only if B**. This means that $A$ is only true if $B$ is true. In other words, if $B$ is not true then $A$ is not true.

4. A **if and only if** B. This means that if $B$ is true then so is $A$, and that if $B$ is not true then $A$ is not true. This is denoted by

$$A \Leftrightarrow B.$$

5. The **converse** to the statement "If A is true then B is true" is the statement "if B is true then A is true".

6. The **contrapositive** to the statement "If A is true then B is true" is the statement if "B is false then A is false".

7. For **some** means for at least one.

8. For **each** means for every one.

**Example 5** *1. Here are **four** statements:*

*(a) Statement A: John and Mary are students in Math 310.*

*(b) Statement B: John and Mary are registered as students in UMD.*

*(c) A implies B.*

*(d) B implies A.*

*In this Example, The statement A implies B is true, and the statement B implies A is false.*

*2. The statement "for some natural number there is a natural number that is smaller" is true, since 1 is less than 2.*

3. *However, the statement "for every natural number there is a natural number that is smaller" is false, since there is no natrural number smaller thatn 1.*

4. *let $S$ be a set, and let*

   (a) *$A$ be the statement " Every $x \in S$ has property $P$, and let*

   (b) *$B$ be the statement " Some $x \in S$ has Property $P$.*

   *Then $A$ implies $B$ but $B$ does not imply $A$.*

**Exercise 3**

1. *If $A \Rightarrow B$ and $B \Rightarrow C$ does $A \Rightarrow C$ ?*

2. *Construct three statements, A,B,C in which A implies B, A implies C, but it is not true that B implies C. Prove that each of your assertions is correct.*

3. *Is the following statement true or false? Provide a proof for your answer. "Suppose $A$ and $B$ are natural numbers, and that $A = 5$ if and only if $B = 2$. If $B \neq 2$, then $A = 7$.*

4. *What is the contrapositive of the following statements?*

   (a) *$A \neq 8$ if $B = 9$.*

   (b) *$A = 8$ if $B \neq 9$.*

5. *What is the converse of the following statements?*

   (a) *$B = 9$ if $A = 8$.*

   (b) *$A \neq 8$ if $B = 9$.*

6. *Show that the statemen "$A \Rightarrow B$" is true if and only if its contrapositive is true.*

## 2.7 Examples of Mathematical Statements and their Proofs

**Example 6**

1. ***Lemma*** *Let $(x_n)$ be the sequence of rational numbers defined by $x_n = 1 - 1/n$. Then for **each** $k \in \boldsymbol{N}$ there is **some** natural number $N$ such that $|1 - x_n| < 1/2k$ if $n \geq N$.*

   *Proof: Fix a natural number $k$.*
   *Then choose $N = 3k$. Thus if $n \geq N$, then $1/n \leq 1/N$, and*

   $$|1 - x_n| = |1 - (1 - 1/n)| = 1/n \leq 1/N = 1/3k < 1/2k.$$

   *q.e.d.*

2. ***This example illustrates the importance of using words carefully and correctly.***

   *Let $(x_n)$ be the sequence of integers defined by $x_n = (-1)^n$. Consider the following two assertions*

   (a) ***Statement:*** *There is an $\varepsilon \in \boldsymbol{R}$, and there is an $N \in \boldsymbol{N}$ such that $|x_n - 0| < \varepsilon$ for $n \geq N$.*

   (b) ***Statement:*** *For each $\varepsilon > 0$ there is an $N \in \boldsymbol{N}$ such that $|x_n - 0| < \varepsilon$ for $n \geq N$.*

   ***The first is correct and the second is false.***

   *To prove the first statement we will pick some $\varepsilon \in \boldsymbol{R}$ and some $N \in \boldsymbol{N}$ so that the conclusion is satisfied. So we pick*

   $$\varepsilon = 3 \quad \text{and} \quad N = 1.$$

   *Then for $n \geq N$,*

   $$|x_n - 0| = |(-1)^n - 0| = 1 < 3 = \varepsilon.$$

   *Now for the second statement. It states that for **each** $\varepsilon$ "something" is true. Thus to prove that it is false we have to find some $\varepsilon$ for which that something is not true.*

   *Here we pick $\varepsilon = 1/2$. Then for any $N \in \boldsymbol{N}$, and $n \geq N$,*

   $$|x_n - 0| = 1$$

   *which is not less than $1/2$.*

3. ***Statement:*** *For each real number $a$ and for each $\varepsilon > 0$ there is a $\delta > 0$ such that if $|x - a| < \delta$ then $|x^2 - a^2| < \varepsilon$.*

   *In this example $\delta$ **will depend on both a and** $\varepsilon$: a larger $a$ and a smaller $\varepsilon$ will require a smaller $\delta$.*

4. ***Statement:****If $x = 2$ and $y = 4$ then $x + y = 6$. This assertion is true.*

   *However, the **converse** assertion: "if $x + y = 6$ then $x = 2$ and $y = 4$" is false, because if $x = 1$ and $y = 5$ then $x + y = 6$. (This is called a **counterexample**)*

5. ***Lemma:*** *Suppose $x$ and $y$ are natural numbers. Then $xy = 2$ **only if** either $x \geq 2$ or $y \geq 2$.*

***Proof:*** *The hypothesis says that if $xy = 2$ then one of $x$ and $y$ must be at least 2,*
*But if neither $x$ nor $y$ is at least 2 then both must be 1.*
*In this case $xy = 1$, contrary to our hypothesis.*
*Thus one of $x, y$ must be at least 2.*
***q.e.d.***

6. *By contrast, the statement $xy = 2$ **if and only if** one of $x \geq 2$ and $y \geq 2$ is false, since it means two things:*

   (a) *If $xy = 2$, then $x \geq 2$ or $y \geq 2$, **AND***

   (b) *If $x \geq 2$ or $y \geq 2$ then $xy = 2$.*

   *But if $x = y = 2$ then $xy = 4$ and so the second statement is not true.*

7. *This example highlights the importance of **"getting the order right"**, with two similar statements:*

   (a) *For each student in this class there is a date on which that student was born.*

   (b) *For each date there is a student in the class who was born on that date.*

   *The first statement is true and the second is false.*

8. ***Lemma:*** *A natural number $n$ is odd **if and only if** it has the form $n = 2k - 1$ for some natural number $k$.*

   ***Proof:*** *If $n$ has the form $n = 2k - 1$ for some natural number $k$, then it is not divisible by 2.*
   *Therefore $n$ is odd.*

   *On the other hand, any natural number, when divided by 2 has a remainder of either 0 or 1.*
   *If the remainder is 0 then the number is divisible by 2, and so it is even.*

   *Thus if $n$ is odd, when divided by 2 it has a remainder of 1.*
   *In this case $n = 2m + 1$ where $m = 0$ or $m \in \boldsymbol{N}$. Set $k = m + 1$. Then $k \in \boldsymbol{N}$ and $n = 2k - 1$.*
   ***q.e.d.***

9. ***Lemma:*** *For every rational number there is a natural number which is larger.*

   ***Proof:*** *The rational number must have the form $p/q$ in which $p \in \boldsymbol{Z}$ and $q \in \boldsymbol{N}$.**(Establishes notation)***
   *If $p \leq 0$ then $p/q < 1$.*

*Otherwise $p \in \boldsymbol{N}$ and $p/q \leq p < p+1$.*
***q.e.d.***

10. ***Lemma:*** *For every positive rational number, $a$, there is a natural number $n$ such that $1/n < a$.*

    ***Proof:*** *Write $a = p/q$ with $p, q \in \boldsymbol{N}$.*
    *Then $1/(q+1) < 1/q \leq p/q$. Set $n = q+1$.*
    ***q.e.d.***

**Exercise 4** 1. *What is the hypothesis and what is the conclusion in the following assertions?*

   (a) *The cube of an odd natural number is odd.*

   (b) *For every $\varepsilon > 0$ there is some $p \in \boldsymbol{N}$ such that $1/p \leq \varepsilon$.*

2. *What is the contrapositive of the statement: For every $\varepsilon > 0$ there is a point $(x, y)$ in the plane whose distance from the origin is between $\varepsilon/2$ and $\varepsilon$.*

3. *What is the converse of the following statement about natural numbers: $n, m$?*

   *If $n > m$ then for some $k \in \boldsymbol{N}, km > n$.*

   *Is the original statement true? Is the converse true? Provide proofs that show your answers are correct.*

4. *What is the converse of the statement "A implies B"? Construct an example where a statement is false but the converse is true.*

5. *What is the converse of the statement "A is true if and only if B is true."? What is the difference in meaning between the original statement and the converse?*

6. *Is (a) or (b) a proof that the following statement is wrong?*
   ***Statement:*** *All ducks are red.*

   (a) *There exists a yellow duck, $A$. Therefore not all ducks are red.*

   (b) *We have not seen all ducks. Therefore some may not be red.*

7. *Prove that if $k \in \boldsymbol{N}$ and $k^2$ is even then $k^2$ is divisible by 4.*

8. *Suppose $x, y$ are positive real numbers and $n \in \boldsymbol{N}$.*

   (a) *Show that $x < y$ if and only if $x^n < y^n$.*

   (b) *If $x < y$ show that if $n > 1$ then $y^n - x^n < n(y-x)y^{n-1}$.*

9. *In each of the following statements, replace the blanks by words/expressions from the list:*

   *let, therefore, if, only if, only, since, therefore, some, then, by, even, implies, because, each, not, true, false, hypothesis, zero, one, theorem, follows*

   *so that the statements are true. Then supply a proof.*

   (a) *___ $n$ is a positive natural number ___ $n+1$ is divisible ___ a prime number.*
   (b) *___ ___ natural number divides a fixed integer then that integer is ___.*
   (c) *___ $S$ be a set with a single element. ___ there is ___ ___ sequence starting at ___ listing elements of $S$.*
   (d) *The statements "A implies B" and "B is ___ true ___ A is ___ true" are equivalent.*
   (e) *___ $x < y$ be real numbers. ___ $y < 0$ ___ $|y| < |x|$.*

10. *In each of the following, replace the blanks by words/expressions from the list in the previous problem to provide a proof of the given statement.*

    (a) ***Statement**: The sum of an even number of odd integers is even.*

    ***Proof:** ___ $k$ be the number of integers and ___ $n_i$ be the $i^{th}$ integer.*
    *___ $k$ is even, $k = 2m$, where $m$ is an integer.*
    *___ $n_i$ is odd, $n_i = 2m_i - 1$, where $m_i$ is an integer.*
    *___*

    $$\sum_{i=1}^{k} n_i = \sum_{i=1}^{2m}(2m_i - 1) = 2\sum_{i=1}^{2m}(m_i) - \sum_{i=1}^{2m} 1 = 2\sum_{i=1}^{2m}(m_i) - 2m.$$

    *___ $\sum_{i=1}^{k} n_i$ is ___.*
    *q.e.d.*

    (b) ***Statement**: If $k \in \boldsymbol{N}$ and $k^2$ is divisible by $3$, then $k^2$ is divisible by $9$.*

    ***Proof:** ___ $r$ be the remainder when $k$ is divided by $3$.*
    *___ $k = 3p + r$ for ___ $p \in \boldsymbol{Z}$.*
    *___ $k^2 = 9p^2 + 6pr + r^2$.*
    *Since by ___ , $k^2$ is divisible by $3$, it follows that $r^2$ is divisible by $3$.*
    *But $r$ is one of $0, 1, 2$ and $1 = 1^2$ and $4 = 2^2$ are not divisible by $3$.*
    *___ $r = 0$ and $k = 3p$ is divisible by $3$.*
    *q.e.d.*

(c) ***Statement:*** *For any natural numbers $p$ and $n$, $(p+1)^n - p^n \geq n$.*

***Proof:*** *By the Difference Theorem,*

$$(p+1)^n - p^n = (p+1-p)\sum_{1=0}^{n-1}(p+1)^i p^{n-i-1} = \sum_{1=0}^{n-1}(p+1)^i p^{n-i-1}.$$

---

$$(p+1)^i \geq 1 \quad and \quad p^{n-i-1} \geq 1,$$

*it* --- *that*

$$\sum_{1=0}^{n-1}(p+1)^i p^{n-i-1} \geq \sum_{1=0}^{n-1} 1 \geq n.$$

--- $(p+1)^n - p^n \geq n$.
*q.e.d.*

## 2.8 The True or False Principle: Negations, Contradictions, and Counterexamples

Central to proofs in Mathematics is the

**Principle: In Mathematics a statement must be be true or false, and may not be both.**

The **negation** of the statement "A is true" is the statement "A is false".

**Example 7** *1. The negation of the negation of a statement A is just the Statement A.*

2. *The negation of the statement: "All integers are even" is the statement "Some integer is odd".*

3. *The negation of the statement: "There is a smallest positive rational number" is the statement "For every positive rational number x there is a positive rational number y such that y¡x.*

4. *The negation of the statement "A is false" is the statement "A is true".*

5. *The negation of the statement "For every $\varepsilon > 0$ there is a $\delta > 0$ such that $\delta > \varepsilon$" is the statement " there is a largest positive real number."*

Thus one way of proving that a statement, A, is true is to show that A cannot be false; i.e., that its negation cannot be true. We can do this using **proof by contradiction**, which works like this:

1. Assume that A is false.

2. Deduce that statement, B, known to be true must then also be false.

3. Since B is known to be true, conclude that A cannot be false.
4. Therefore A must be true.

Here are some examples of this method:

**Example 8** ***Proof by contradiction***

1. ***Prove that there are infinitely many prime numbers.***

   ***Proof:*** *Suppose the statement "there are infinitely many prime numbers" is false. Then there would be only finitely many prime numbers. Thus there would be a largest prime number $p$.*

   *Let $n$ be the number obtained by first multiplying all the natural numbers from 1 to $p$ and then adding 1. (**Establishes notation**) Then $n > p$.*

   *Dividing $n$ by any natural number $k \leq p$ gives a remainder 1. Therefore $n$ is not divisible by any natural number $k \leq p$.*

   *But $n$ must be divisible by some prime number $q$. Therefore there must be a prime number $q > p$. This contradicts the hypothesis that $p$ was the largest prime number. Therefore there must be infinitely many primes.* ***q.e.d.***

   ***Remark:*** *This proof was discovered by Euclid about 300 BC.*

2. ***Prove that there is not a smallest positive rational number.***

   ***Proof:*** *Suppose the statement "there is not a smallest positive rational number" is false. Then there is a smallest positive rational number, $x$.*

   *Since $x$ is a positive rational number, it must be of the form $p/q$, with $p, q \in \boldsymbol{N}$. But*

   $$p/(q+1) - p/q = \frac{pq - p(q+1)}{q(q+1)} = -\frac{p}{q(q+1)} < 0.$$

   *Thus*

   $$p/(q+1 < p/q,$$

   *and so $p/q$ is not the smallest positive rational number.*

   *This is a contradiction, and proves that there is not a smallest positive rational number.* ***q.e.d.***

**Exercise 5** ***Proof by contradiction***

1. *Show that a statement is true if and only if its contrapositive is true.*

2. *Can a statement and its negation both be true? Justify your answer.*

3. *Show that there is not a largest even natural number.*

4. *Show that among the rational numbers of the form $\frac{p}{p+1}$ with $p \in \boldsymbol{N}$ there is not a largest one.*

5. *Show that there is not a largest negative rational number.*

6. *Prove that 2 is not the square of a rational number.*

**Counterexamples are an important way to show that a statement is false**. Many mathematical statements have the form

"If A is true then B is true".

To show that this assertion is false it is enough to find a simple example in which A is true but B is not true. Such an example is called a **counterexample**.

**Example 9** *Counterexamples*

1. *To prove that the statement **"all natural numbers are prime"** is false you only have to find a **single** example (eg. $6 = 2 \cdot 3$).*

2. *The statement that all students in Math 310 are seniors is false: there are several counterexamples in this class.*

3. *There is a much deeper example from the theory of equations. You learned in school that the quadratic equation*

$$ax^2 + bx + c = 0$$

*has two solutions, $x = -b \pm \sqrt{b^2 - 4ac})/2a$. One might ask whether the solutions to higher order equations can also be expressed in formulas that only use $n^{th}$ roots, for some natural numbers $n$, and indeed this is true for cubic and quartic equations.*

*However, it is not true for all fifth degree equations, as was shown by Abel and Ruffini in 1824. The easiest way to show this is by an explicit counterexample and a very simple one was proved about a century ago by a famous algebraist, Emil Artin, who showed that roots of the equation $x^5 - x - 1 = 0$ cannot be expressed in this way.*

**Exercise 6** ***Counterexamples***

1. *Show that the following statement is false: If $a, b \in \boldsymbol{N}$ then $a^3 + b^3$ is the cube of a natural number.*

2. *Prove that the sum of an odd number of natural numbers is odd if each of the natural numbers is odd. What is the converse statement? Show by a counterexample that the converse is false.*

3. *What is the converse to the statement: "If $n$ is an odd natural number then $n^2$ is odd". Decide if it is true or false and prove your answer.*

4. *Prove or disprove the following statement: For every $x \in \boldsymbol{Q}$ there is a unique $n \in \boldsymbol{N}$ which is the closest natural number to $x$.*

5. *Construct an example to show that if in each of two school classes the average GPA of the boys is bigger than that of the girls it may not be the case that when the classes are combined this is still true: i.e., in the combined classes the average GPA of the girls may be bigger than that of the boys.*

6. *Show by counterexample that the following statement is false: "For every $p, q \in \boldsymbol{N},\ p^2 + q^2 > (p+q)^2 - 50p - 51q$.*

7. *Show by counterexample that the following statement is false: "If $t, u, s \in \boldsymbol{N}$ satisfy $s > 39, t > s + 5$, and $u > 33$, then $t + u > 82$.*

## 2.9 Proof and Construction by Induction

**Induction is used to prove that every statement $S(n)$ in an infinite sequence of statements is true.** It is based on the following induction principle, which we will take as true without proof!

**Induction Principle:** Suppose given a sequence of statements $S(n)$, one for each $n \in \mathbf{N}$. Suppose that

1. $S(1)$ is true.
2. Whenever $S(n)$ is true then also $S(n+1)$ is true.

Then $S(n)$ is true for all $n \in \mathbf{N}$.

**Example 10** ***Examples of proof by induction***

1. ***Statement:*** *If $n, m \in \boldsymbol{N}$ and $x$ is a non-zero real number then*

$$x^{n+m} = x^n x^m.$$

***Proof:*** *We prove this by induction on $n$, and set*

$$S(n): \text{ For every } m,\ x^{n+m} = x^n x^m$$

*Then S(1) reads for every $n \in \boldsymbol{N}$, $x^{n+1} = x^n x$, which is true by definition.*

*Now suppose $S(n)$ is true for some $n$. Then because $S(n)$ is true,*

$$x^{(n+1)+m} = x^{n+1+m} = x^n x^{1+m} = x^n x^m x = x^n x x^m = x^{n+1} x^m.$$

*Thus $S(n+1)$ is true, and so by induction, $S(n)$ is true for all $n$.* ***q.e.d.***

2. ***Statement:*** *If $n, m \in \boldsymbol{N}$ and $x$ is a non-zero real number then*

$$(x^n)^m = x^{nm}.$$

***Proof:*** *We prove this by induction on $m$, and set*

$$S(m) : (x^n)^m = x^{nm} \quad \textit{for every } n \in \boldsymbol{N}, \quad .$$

*Then S(1) reads for any $n \in \boldsymbol{N}$, $(x^n)^1 = x^n$, which is true by definition. Now suppose $S(m)$ is true for some $m$. Then because $S(m)$ is true, we may use the Statement above to obtain*

$$(x^n)^{m+1} = (x^n)^m x^n = x^{nm} x^n = x^{nm+n} = x^{n(m+1)}.$$

*Therefore $S(m+1)$ is true, and so by induction $S(m)$ is true for all $m$.* ***q.e.d.***

3. ***Statement:*** *For every $n \in \boldsymbol{N}$,*

$$\sum_{k=1}^{k=n} k^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

***Proof:*** *We prove this by induction on $n$, with $S(n)$ the equation above.*

*When $n = 1$ both sides of the equation equal 1 and so they equal each other.*

*Suppose now by induction that $S(n)$ is true for some $n$.*

*Then for this $n$, since $S(n)$ is assumed to be true,*

$$\sum_{k=1}^{k=n+1} k^3 = \sum_{k=1}^{k=n} k^3 + (n+1)^3 = \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3.$$

*Factoring out $(n+1)^2$ gives*

$$\sum_{k=1}^{k=n+1} k^3 = (n+1)^2(\frac{n^2}{4}+n+1) = (n+1)^2(\frac{n^2+4n+4}{4}) = \left(\frac{(n+1)(n+2)}{2}\right)^2.$$

*Therefore $S(n+1)$ is true, and the formula follows by induction.* ***q.e.d.***

4. ***Statement:*** *For any $n \in \boldsymbol{N}$,*

$$\sum_{k=1}^{n} k \leq \sum_{k=1}^{n} k^2.$$

***Proof:*** *When $n = 1$ the inequality reduces to $1 \leq 1$, which is true. Suppose the inequality holds for some $n$.*
*Observe that $(n+1)^2 = (n+1)(n+1) = n^2 + 2n + 1 \geq n + 1$.*
*Therefore*

$$\sum_{k=1}^{n+1} k = \sum_{k=1}^{n} k + (n+1) \leq \sum_{k=1}^{n} k^2 + (n+1)^2 = \sum_{k=1}^{n+1} k^2.$$

*Therefore the inequality holds for $n+1$, and so by induction the inequality holds for all $n$.* ***q.e.d.***

An important application of proof by induction is the **Binomial Theorem**, which requires a definition and some more notation.

**Definition 4** *1. For any natural number $n$,* ***n factorial****, denoted* $\mathbf{n}!$*, is the product of all the natural numbers from 1 to $n$:*

$$n! = \Pi_{k=1}^{n} k.$$

*Additionally we define 0 factorial to be 1: $0! = 1$.*

*2. We write*

$$\begin{pmatrix} n \\ i \end{pmatrix} = \frac{n!}{i!(n-i)!}$$

*These numbers are called* ***binomial coeeficients.***

Now we can state the:

**Theorem 2** ***(Binomial theorem)*** *For any real numbers $a$ and $b$, and for any $n \in \boldsymbol{N}$,*

$$(a+b)^n = \sum_{i=0}^{n} \begin{pmatrix} n \\ i \end{pmatrix} a^i b^{n-i}.$$

To prove this we first need to establish an important property of the binomial coeeficients:

**Lemma 1** *For any natural number, $n$, and any natural number $i \leq n+1$,*

$$\begin{pmatrix} n \\ i-1 \end{pmatrix} + \begin{pmatrix} n \\ i \end{pmatrix} = \begin{pmatrix} n+1 \\ i \end{pmatrix}.$$

**Proof:** First, we use the definition above to rewrite the left hand side as

$$\frac{n!}{(i-1)!(n-(i-1))!} + \frac{n!}{i!(n-i)!}.$$

Multiply the numerator and the denominator of the first term by $i$ to get

$$\frac{n!}{(i-1)!(n-(i-1))!} = \frac{n! \cdot i}{i!(n+1-i)!}.$$

Then multiply the numerator and the denominator of the second term by $n+1-i$ to get

$$\frac{n!}{i!(n-i)!} = \frac{n! \cdot (n+1-i)}{i!(n+1-i)!}.$$

Finally, add together to get

$$\frac{n!}{(i-1)!(n-(i-1))!} + \frac{n!}{i!(n-i)!} = \frac{n! \cdot (i+n+1-i)}{(i)!(n+1-i)!} = \frac{(n+1)!}{(i)!(n+1-i)!}.$$

**q.e.d.**

**Proof of the binomial theorem:** We prove this by induction on $n$.
When $n = 1$ the statement reduces to $a + b = a + b$.
Next, assume the statement is true for $n$. Then it follows that

$$(a+b)^{n+1} = (a+b)(a+b)^n = (a+b)\sum_{i=0}^{n} \left( \begin{array}{c} n \\ i \end{array} \right) a^i b^{n-i}\ .$$

Multiplying separately by $a$ and $b$ and adding, we get

$$(a+b)^{n+1} = \sum_{i=0}^{n} \left( \begin{array}{c} n \\ i \end{array} \right) a^{i+1} b^{n-i} + \sum_{i=0}^{n} \left( \begin{array}{c} n \\ i \end{array} \right) a^i b^{n-i+1}.$$

We consider the two terms separately. The first term may be written as

$$\sum_{i=0}^{n} \left( \begin{array}{c} n \\ i \end{array} \right) a^{i+1} b^{n-i} = \sum_{i=0}^{n-1} \left( \begin{array}{c} n \\ i \end{array} \right) a^{i+1} b^{n-i} + a^{n+1}.$$

For this first term set $k = i + 1$. Then as $i$ runs from 0 to $n-1$, $k$ runs from 1 to $n$, and

$$n - i = (n+1) - (i+1) = n + 1 - k.$$

Thus the first term may be rewritten as

$$\sum_{k=1}^{n} \left( \begin{array}{c} n \\ k-1 \end{array} \right) a^k b^{n+1-k} + a^{n+1}.$$

Next note that here, $k$ is a "dummy variable". We could have used any symbol such as $m, n, p, \ldots$. In particular we could have used $i$. Thus this term may be rewritten as

$$\sum_{i=1}^{n} \left( \begin{array}{c} n \\ i-1 \end{array} \right) a^i b^{n+1-i} + a^{n+1}.$$

On the other hand, the second term may be written as

$$\sum_{i=0}^{n} \left( \begin{array}{c} n \\ i \end{array} \right) a^i b^{n-i+1} = b^{n+1} + \sum_{i=1}^{n} \left( \begin{array}{c} n \\ i \end{array} \right) a^i b^{n-i+1}.$$

Therefore, adding the two terms together we get

$$(a+b)^{n+1} = a^{n+1} + \Sigma_{i=1}^{n} \left( \left( \begin{array}{c} n \\ i-1 \end{array} \right) + \left( \begin{array}{c} n \\ i \end{array} \right) \right) a^i b^{n+1-i} + b^{n+1}.$$

Hence we may apply Lemma 1 to conclude that

$$(a+b)^{n+1} = \sum_{i=0}^{n+1} \left( \begin{array}{c} n+1 \\ i \end{array} \right) a^i b^{n+1-i}.$$

This shows that the statement is true for $n+1$, and so by induction it is true for all $n$. **q.e.d.**

There is another form of the induction principle we shall also use, as illustrated in the next Proposition:

**Proposition 2** *Suppose given a sequence of statements $S(n)$, one for each $n \in \boldsymbol{N}$, and that*

1. *$S(1)$ is true.*
2. *Whenever $S(k)$ is true for $k \leq n$ then also $S(n+1)$ is true.*

*Then $S(n)$ is true for all $n \in \boldsymbol{N}$.*

**Proof:** Let $T(n)$ be the statement: $S(k)$ is true for all $k \leq n$.
Since $S(1)$ is true, so is $T(1)$.
Suppose by induction that $T(n)$ is true.
Then $S(k)$ is true for all $k \leq n$.
Therefore by hypothesis, $S(n+1)$ is true.
Since $S(k)$ is true for $k \leq n$ as well, it is true for $k \leq n+1$.
Thus $T(n+1)$ is true.
Now it follows from the induction principle that $T(n)$ is true for all $n$.
Therefore $S(n)$ is true for all $n$. **q.e.d.**

**Induction is also used to construct infinite sequences.** This method is based on the

**Construction by Induction Principle** To construct a sequence $(x_n)_{n\geq 1}$ it is sufficient to

1. First, construct $x_1$.

2. Then, assuming that $x_i$ has been constructed for $i \leq n$, give some explicit construction for $x_{n+1}$.

The **induction construction principle** then states that this constructs a specific infinite sequence $(x_n)_{n\geq 1}$ **defined for all** $n$**.**

**Example 11** ***Construction by induction***

1. *An infinite sequence $(x_n)_{n\geq 1}$ is constructed by induction as follows:*
   (a) *Set $x_1 = 2$.*
   (b) *Assume $x_i$ is constructed for $i \leq n$ and set*
   $$x_{n+1} = \sum_{i=1}^{n} (-1)^i x_i.$$
2. *An infinite sequence $(x_n)_{n\geq 1}$ satisfying $x_{n+1} > x_n^2$ is constructed by induction as follows:*
   (a) *Set $x_1 = 1$*
   (b) *Assume $x_i$ is constructed for $i \leq n$ and set*
   $$x_{n+1} = x_n^2 + 1.$$
3. *An infinite sequence of natural numbers $(a_k)_{k\geq 1}$ is constructed by induction by setting*
   $$a_1 = 1, \quad \textit{and}$$
   $$a_{k+1} = \begin{cases} \frac{\sum_{i=1}^{i=k} a_i}{2}, & \textit{if } \sum_{i=1}^{i=k} a_i \textit{ is even;} \\ k, & \textit{if } \sum_{i=1}^{i=k} a_i \textit{ is odd.} \end{cases}$$

**Exercise 7** ***Induction***

1. *Show by induction that $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$.*
2. *Show by induction that $\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.*
3. *Use the Binomial theorem to show that if $p, k \in \boldsymbol{N}$ then*
   $$(\frac{p+1}{p})^k > \frac{k}{p}.$$
   *Conclude that*
   $$(\frac{p}{p+1})^k < \frac{p}{k}.$$
4. *Show by induction on $n \in \boldsymbol{N}$ that the product of $n$ odd natural numbers is odd.*

5. *Let $p_1 < ... < p_n < ...$ be an infinite sequence of prime numbers with $p_1 = 5$. Show by induction that $p_n > 2n$.*

6. *Construct by induction an infinite sequence of natural numbers $(x_n)_{n\geq 1}$ such that for $n \geq 2$, $x_n > \sum_{i=1}^{n-1} x_i$.*

7. *Construct by induction an infinite sequence of rational numbers $(a_n)_{n\geq 1}$ such that for all $n, a_n > a_{n+1} > 0$, and $\Sigma_{i=1}^{n} a_i < 1$.*

## 2.10 Polynomials

**Definition 5**

1. *A* ***polynomial*** *of degree $n$ is a complex-valued function defined in* $\boldsymbol{R}$ *of the form $P(x) = \Sigma_{k=0}^{n} a_k x^k$, in which each $a_k \in \boldsymbol{C}$ and $a_n \neq 0$.*

2. *The $a_k$ are called the* ***coefficients*** *and $a_n$ is the* ***leading*** *coefficient.*

3. *If the coefficients are all real numbers then $P(x)$ is a* ***real polynomial****.*

4. *If the coefficients are all zero then the polynomial has degree -1.*

Polynomials play key roles in algebra and analysis. For example (but well beyond the scope of this course!), if $f(x)$ is any continuous function defined in an interval $[a, b]$ then for any positive number $\varepsilon$ there is a real polynomial $P(x)$ such that

$$|f(x) - P(x)| < \varepsilon \text{ for all } \quad x \in [a, b].$$

In algebra we can add and multiply polynomials:

$$\Sigma a_k x^k + \Sigma b_k x^k = \Sigma(a_k + b_k)x^k,$$

and

$$(\Sigma a_k x^k)(\Sigma b_k x^k) = \Sigma_i(\Sigma_{k+l=i} a_k b_l)x^i.$$

Also, for polynomials, similar to the Division Proposition for Numbers, there is the

**Proposition 3** *The Division Proposition for Polynomials*

*Suppose $P(x)$ is a polynomial of degree $n$, and $D(x)$ is a polynomial of degree $m$. If $n \geq m$, then*

$$P(x) = D(x)Q(x) + R(x),$$

*where $Q(x)$ is a polynomial of degree n-m and $R(x)$ is a polynomial of degree $r < m$.*

**Proof:** We use induction on $n - m$.
Denote the leading coefficients of $P(x)$ and of $D(x)$ respectively by $a_n$ and by $b_m$, and define a polynomial $S(x)$ by

$$S(x) = \begin{cases} P(x) - \frac{a_n}{b_m} x^{n-m} D(x) & \text{if } n > m \\ P(x) - \frac{a_n}{b_m} D(x) & \text{if } n = m \end{cases}$$

By construction: $S(x)$ has the form

$$S(x) = P(x) - Q_1(x)D(x) \text{ with } degS(x) = l < n$$

Suppose first that $l \geq m$. Since $l - m < n - m$, by our induction hypothesis,

$$S(x) = Q_2(x)D(x) + R(x) \text{ with } degR(x) < m.$$

Therefore

$$P(x) = S(x) + Q_1(x)D(x) = P(x) + (Q_1(x) + Q_2(x))D(x) + R(x).$$

On the other hand, if $l < m$ then we may choose $R(x) = S(x)$, since

$$P(x) = Q_1(x)D(x) + S(x)$$

q.e.d.

**Exercise 8** 1. *if $P(x)$ is the product of two polynomials $P_1(x)$ and $P_2(x)$, show that deg $P(x) = degP_1(x) + degP_2(x)$.*

2. (a) *A root of a polynomial $P(x)$ is a number $c$ such that $P(c) = 0$. Show that $c$ is a root of $P(x)$ if and only if $P(x)$ =$(x - c)Q(x)$ for some polynomial $Q(x)$.*
   (b) *Using this fact show that a polynomial of degree $n$ has at most $n$ distinct roots.*
   (c) *If a polynomial $P(x)$ of degree $n$ does have $n$ distinct roots, use the same fact to show that it has the form $P(x) = a_n \Pi_{i=1}^{n}(x - c_i)$.*

## 2.11 The Literature of Mathematics

Mathematicians have built up a body of knowledge over several thousand years, expressed in theorems and examples, all validated by rigorous proofs, and that once validated are then true forever. This is the *literature of mathematics.* Each new piece of knowledge depends on what came before, and sometimes it takes many decades for little pieces to fit together to establish some remarkable new phenomenon.

The classical example is 'Fermat's last theorem'! Fermat was a 17th century number theorist, and after he died in 1637 the following statement was found written in the margin of one of his books: "I have found the most wonderful result, but the margin is too small for me to write down the proof." The simple assertion was this: If $a, b, c, n$ are natural numbers all greater than 1, and if

$$a^n + b^n = c^n,$$

then $n = 2$.

In the following three hundred years the search for a proof was one of the 'holy grails' of mathematics. Much of our knowledge in number theory and geometry was developed in the process. And finally in 1995 Andrew Wiles (Princeton) published a proof using many of the results which had been established over the preceding centuries.

Some of you may wonder about the relation of mathematics to the physical world we live in. Now mathematical knowledge itself is validated just by proofs, and so its correctness stands for all time and does not depend in any way on our physical experience. Nonetheless, much of mathematics is inspired by that physical experience, mathematical knowledge is regularly applied in almost every branch of science and engineering, and indeed many mathematicians focus on building mathematical models and computational algorithms that directly reflect/compute physical phenomena. As a Nobel Laureate in Physics, Eugene Wigner, wrote in 1960 in a paper entitled *The Unreasonable Effectiveness of Mathematics* "...the mathematical formulation of the physicist's often crude experience leads in an uncanny number of cases to an amazingly accurate description of a large class of phenomena".

# Chapter 3

# Basic Set Theory

## 3.1 Sets

Sets, and maps between sets, are part of the basic language of Mathematics in many areas, including algebra, analysis, logic and topology. **In particular, the material in this chapter will be used throughout the rest of this course!**

**Definition 6** ***Sets***

1. *As stated in Section 2.4, a* **set** *is any* ***explicitly*** *specified collection of objects, abstract or concrete. If $S$ is a set, we sometimes write*

$$S = \{\cdots\},$$

   *where the elements of the set are listed between the parentheses.*

2. *The 'objects' are called the* **elements** *of the set.*

3. *If $x$ is an element in a set $S$ we say $x$* **belongs** *to $S$, and denote this by $x \in S$.*

4. *A* **finite** *set is a set with only finitely many elements. In this case $|S|$ denotes the number of elements in $S$.*

5. *A set which is not finite is called* ***infinite***.

6. *The* **empty set** *is the set with no elements. It is denoted by $\phi$.*

**Note: In the exercises, in order to define a set, you must specify its elements.**

**Example 12** *Sets*

1. *The set* $S = \{1, 2, 3\}$ *has three elements:* $1, 2,$ *and* $3$ *and so* $|S| = 3$.
2. *The empty set* $\phi$ *has no elements and so* $|\phi| = 0$.
3. *Each of* $\boldsymbol{N}$, $\boldsymbol{Q}$, $\boldsymbol{R}$, *and* $\boldsymbol{C}$ *are* ***infinite sets****.*
4. *The unit interval of all real numbers* $x$ *satisfying* $0 \leq x \leq 1$ *is an infinite set.*
5. *The desks in this classroom are a finite set.*
6. *The real numbers whose squares are natural numbers are a set.*

## 3.2 Operations with Sets

**Definition 7** ***Subsets***

*A* **subset** *of a set* $S$ *is a set* $W$ ***all*** *of whose elements are elements of* $S$*. We denote this by* $W \subset S$ *and we often write*

$$W = \{x \in S \mid \cdots\},$$

*where* $\cdots$ *specifies which elements of* $S$ *are in* $W$*.*

**Example 13** ***Subsets***

1. *The set* $S$ *of even integers is the subset of* $\boldsymbol{Z}$ *given by*

   $$S = \{n \in \boldsymbol{Z} \mid n \text{ is divisible by } 2\}.$$

   *It is* ***not*** *a finite set.*
2. $\boldsymbol{N}$ *is the subset of* $\boldsymbol{Z}$ *defined by*

   $$\boldsymbol{N} = \{n \in \boldsymbol{Z} \mid n > 0\ \}.$$

3. $\boldsymbol{Q}$ *is the subset of* $\boldsymbol{R}$ *given by*

   $$\boldsymbol{Q} = \{x \in \boldsymbol{R} \mid x = p/q \text{ with } p \in \boldsymbol{Z} \text{ and } q \in \boldsymbol{N}\}.$$

4. *The empty set is a subset of every set.*

5. *The intervals* $[c,d],[c,a),(a,b),$ *and* $(b,d]$ *are the subsets of* $\boldsymbol{R}$ *given by*

$$[c,d]=\{x\in\boldsymbol{R}\mid c\le x\le d\},\ [c,a)=\{x\in\boldsymbol{R}\mid c\le x<a\},\ (a,b)=\{x\in\boldsymbol{R}\mid a<x<b\},\quad \textit{and,}$$

$$(b.d]=\{x\in\boldsymbol{R}\mid b<x\le d\}.$$

6. *Let* $L$ *be the set whose elements are all the lines in the plane. Thus a line in the plane is a subset of the plane, and is also an element in* $L$.

7. *The vertical lines in the plane are a subset of* $L$ .

**Definition 8** ***Unions and intersections***

*Suppose* $S_\alpha$ *is a collection of sets. Then*

1. *The* ***union*** *of the* $S_\alpha$ *is the set*

$$\bigcup_\alpha S_\alpha=\{x\mid x\in S_\alpha \textit{ for } \textbf{\textit{some}}\ S_\alpha\}.$$

*In words:* $\bigcup_\alpha S_\alpha$ *is the set whose elements are those objects which are elements in* ***at least one*** *of the* $S_\alpha$.

2. *The* ***intersection*** *of the* $S_\alpha$ *is the set*

$$\bigcap_\alpha S_\alpha=\{x\mid x\in S_\alpha \textit{ for } \textbf{\textit{each}}\ \alpha\}.$$

*In words,* $\bigcap_\alpha S_\alpha$ *is the set whose elements are those objects which belong to* ***each*** *of the* $S_\alpha$

3. *Two sets* $S$ *and* $T$ *are* ***disjoint*** *if* $S\cap T=\phi$ *Thus* $S$ *and* $T$ *are disjoint if the have* ***no*** *elements in common.*

**Example 14** ***Unions and intersections***

1. *The union of the sets* $\{1,2,3\}$ *and* $\{2,4,8\}$ *is* $\{1,2,3,4,8\}$. *The intersection of these sets is* $\{2\}$. *The original two sets each have three elements, the union has five elements, and the intersection has one element.*

2. *The union of the sets* $\{1,2\},\{2,3\}$ *and* $\{3,4\}$ *is the set* $\{1,2,3,4\}$. *The intersection of the first three sets is* $\phi$.

3. *Let* $S$ *be the set of even integers and let* $T$ *be the set of odd integers. Then*

$$S\cup T=\boldsymbol{Z}\quad \textit{and}\quad S\cap T=\phi.$$

4. *For each* $n\in\boldsymbol{N}$ *let* $S_n\subset\boldsymbol{N}$ *be the subset of all natural numbers except* $n$. *Then*

*(a)*

$$\bigcup_n S_n = \boldsymbol{N},$$

*since every natural number is in some (in fact in almost all) $S_n$.*

*(b)*

$$\bigcap_n S_n = \phi,$$

*because any natural number $k$ is not $S_k$ and so is not in every $S_n$.*

**Definition 9** ***Products***

*The* ***product*** *of two sets $S$ and $T$ is the set $S \times T$ whose elements are the ordered pairs $(x, y)$ with $x \in S$ and $y \in T$. This is written as*

$$S \times T = \{(x, y) \mid x \in S \text{ and } y \in T\}.$$

**Example 15** ***Products***

1. *The product of the sets $S = \{2, 17, 99\}$ and $T = \{5, \sqrt{2}\}$ is the set*

$$S \times T = \{(2, 5), (17, 5), (99, 5), (2, \sqrt{2}), (17, \sqrt{2}), (99, \sqrt{2})\}.$$

   *Thus in this example, $|S| = 3, |T| = 2$ and $|S \times T| = 6$.*

2. *The product $\boldsymbol{R} \times \boldsymbol{R}$ is the set $\boldsymbol{R}^2$, of points in the plane:*

$$\boldsymbol{R}^2 = \{(x, y) | \ x \in \boldsymbol{R} \text{ and } y \in \boldsymbol{R}\}.$$

3. *The product of $\boldsymbol{Q}$ with $\boldsymbol{R}$ is the set $\{a, y) | \ a \in \boldsymbol{Q}, y \in \boldsymbol{R}\}$ of points in the plane whose x-coordinate is rational. It is a subset of $\boldsymbol{R}^2$.*

4. *The set $\boldsymbol{Q}$ is the set of real numbers of the form $p/q$ with $p \in \boldsymbol{Z}$ and $q \in \boldsymbol{N}$. This is* ***not*** *the product*

$$\boldsymbol{Z} \times \boldsymbol{N} = \{p, q) | \ p \in \boldsymbol{Z}, q \in \boldsymbol{N}\},$$

   *because $(2, 1)$ and $(4, 2)$ are different elements in the product but $2/1 = 4/2$.*

**Definition 10** *The* ***power set*** *$P(S)$ is the set whose elements are all the subsets of $S$.*

**Example 16** *Power sets*

1. *For any set $S$, $S \in P(S)$ and $\phi \in P(S)$ because $S$ and $\phi$ are subsets of $S$.*

2. *The power set $P(\{1, 2\})$ is given by*

$$P(\{1, 2\}) = \{\phi, \{1\}, \{2\}, \{1, 2\}\}.$$

3. *The power set $P(\phi)$ of $\phi$ is the set*

$$P(\phi) = \{\phi\}.$$

**Exercise 9** *Sets*

1. *Let $S$ be the set of all even integers, and let $T$ be the set of all integers divisible by* 3.
   (a) *What is $S \cap T$ ?*
   (b) *Let $W$ be the set of integers which are not in $S \cup T$. Is $W$ infinite or finite?*
2. *Suppose $S_n = [n, n+1], n \in \boldsymbol{N}$. Find $\bigcup S_n$ and $\bigcap S_n$.*
3. *Suppose $S_n = [1/n, 1], n \in \boldsymbol{N}$. Find $\bigcup S_n$ and $\bigcap S_n$*
4. *Suppose $S$ and $T$ are sets and $|S| = 3$ and $|T| = 4$*
   (a) *What is $|S \times T|$?*
   (b) *What is $|P(S)|$?*
5. *Suppose $S$ and $T$ are sets.*
   (a) *If $S$ and $T$ are disjoint sets show that for any set $W$, $S \times W$ and $T \times W$ are disjoint.*
   (b) *If $S$ and $T$ are disjoint sets for which sets $W$ are $S \cup W$ and $T \cup W$ disjoint?*
   (c) *If $S$ and $T$ are disjoint sets show that $P(S) \cap P(T) = \{\phi\}$.*
6. *If $S$ and $T$ are finite sets show that*

$$|S \cup T| + |S \cap T| = |S| + |T|.$$

7. *Let $S$ be a finite set and suppose $x \notin S$. Show that*

$$|P(S \cup \{x\})| = 2|P(S)|$$

8. *Use the previous problem and induction to prove that if $S$ is a finite set then $|P(S)| = 2^{|S|}$.*

## 3.3 Maps between Sets

**Definition 11** ***Maps***

*A* **map** *$\varphi$ from a set $S$ to a set $T$ consists of* ***three*** *things:*

1. *A set $S$, called the* ***domain*** *of the map;*

2. *A set $T$, called the* ***target*** *of the map; and*

3. *A* ***rule*** *which assigns to each element $x \in S$ a* ***single*** *specified element $y \in T$. The element $y$ is called the* ***image*** *of $x$ and is denoted by $y = \varphi(x)$.*

*This map is often denoted by*

$$\varphi : S \to T.$$

**Definition 12** ***More on maps***

1. *If $\varphi : S \to T$ is a map between sets then the* ***image*** *of $\varphi$ is the subset $Im\ \varphi \subset T$ defined by*

$$Im\ \varphi = \{\varphi(x) \mid x \in S\}.$$

   ***Note:*** *The image of a map is a subset of the target, but it may not be all of the target!*

2. *If $\varphi : S \to T$ is a map, and if $U \subset T$ is a subset then we write*

$$\varphi(U) = \{y \in T \mid y = \varphi(x) \quad \textit{for some} \quad x \in U\}.$$

   *In particular, $Im\ \varphi = \varphi(S)$.*

3. *If $S$ and $T$ are sets, then the collection of all maps from $S$ to $T$ is a set. It is denoted by $T^S$:*

$$T^S = \{\varphi \mid \varphi \textit{ is a map from } S \textit{ to } T\}$$

**Note:**

1. To define a map we must specify **three** things: the domain, the target, and the rule - see Example 17, below.

2. **Do not confuse** a **set**, $S$ with a **map**, $\varphi$.

   (a) A **set** is a collection of objects.

   (b) A **map**, $\varphi : S \to T$ ia a rule connecting each element $x \in S$ to a single element $\varphi(x) \in T$.

**Example 17** ***Maps***

1. *A map $\varphi : \boldsymbol{Z} \to \boldsymbol{Z}$ is defined by*

$$\varphi(a) = \begin{cases} -a, & a \leq 0 \\ a/2, & a \in \boldsymbol{N} \textit{ is even} \\ 0, & a \in \boldsymbol{N} \textit{ is odd.} \end{cases}$$

   ***Maps are often described using the notation of this example.***

2. ***Non-map vs map***

   (a) *The "rule" which assigns*

   $$1 :\rightarrow 1 \textit{ and } 2$$

   *and*

   $$2 :\rightarrow 2$$

   *is **not a map** from* $\{1, 2\}$ *to* $\{1, 2\}$ *because two different elements are assigned to* 1.

   (b) *However, the rule which assigns* 1 *and* 2 *to* 1 *is a map.*

3. *A map* $\varphi : \boldsymbol{N} \rightarrow \boldsymbol{N}$ *is defined by*

   $$\varphi(n) = 2n.$$

   *Here the target is* $\boldsymbol{N}$*, but the image is the subset of even natural numbers.*

4. *Three maps are defined as follows:*

   (a)
   $$\varphi : \boldsymbol{N} \rightarrow \boldsymbol{N};\ \varphi(n) = n^2 + 1.$$

   (b)
   $$\psi : \boldsymbol{Z} \rightarrow \boldsymbol{N};\ \psi(n) = n^2 + 1.$$

   (c)
   $$\chi : \boldsymbol{N} \rightarrow \boldsymbol{Z};\ \chi(n) = n^2 + 1.$$

   *These three maps are all **different even though the "formula" is the same**, because the two sets are different in each case.*

5. *Not all maps can be expressed by a formula. For example, a map* $\varphi : \boldsymbol{N} \rightarrow \boldsymbol{N}$ *is defined by*

   $$\varphi(n) \textit{is the nth largest prime number.}$$

   *This map does not have a formula. In this example,* $\varphi(\boldsymbol{N})$ *is the set of prime numbers.*

6. *A polynomial (defined in Sec.2.10) is a map from* $\boldsymbol{R}$ *to* $\boldsymbol{C}$*. A real polynomial is a map from* $\boldsymbol{R}$ *to* $\boldsymbol{R}$*.*

7. *More generally, a map from* $\boldsymbol{R}$ *to* $\boldsymbol{R}$ *is often called a* real valued function*, and the formulas you are used to from calculus usually define real valued functions on* $\boldsymbol{R}$*. However, **not every real valued function has a formula**.*

8. *An nxn matrix $A = (a_{ij})$ defines the map from the set $\boldsymbol{R}^n$ of column vectors to itself, given by*

$$A \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} \sum_j a_{ij} v_j \\ \vdots \\ \sum_j a_{nj} v_j \end{pmatrix}$$

.

9. *A map $f$ from $\boldsymbol{R}$ to $\boldsymbol{R}$ is defined by $f(x) = x^2$, and a map $g$ from $\boldsymbol{R}$ to the set $T$ of all non-negative real numbers defined by $g(x) = x^2$.*
   ***Note:*** *as above, $f$ and $g$ are* ***different*** *maps even though the formula is the same.*

10. *Suppose $S$ is the set of students enrolled in the University of Maryland on Oct. 1, 2016. Then a map $\varphi : S \to \boldsymbol{N}$ is defined by the rule which assigns to each student the least number of inches which is greater than or equal to their height.*

11. *$S$ is the set of students enrolled in the University of Maryland on Oct. 1, 2016, $C$ is the set of all countries that existed prior to Oct. 1, 2016, and $\varphi$ is the rule which assigns to each student the country in which they were born.*

12. *$\psi : \boldsymbol{N} \to \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ assigns to each $n \in \boldsymbol{N}$ the $n^{th}$ integer in the decimal expansion of $\pi$.*

## 3.4 Composites, the Identity Map, and Associativity

**Definition 13** ***Composites***

*If $\varphi : S \to T$ and $\psi : T \to W$ are maps then their* ***composite*** *is the map $\psi \circ \varphi : S \to W$ defined by*

$$(\psi \circ \varphi)(x) = \psi(\varphi(x)), \quad x \in S.$$

**Important Note:** Composites can be defined more generally, but in this course we shall only consider the situation described above!

**Example 18** ***Composites***

1. *The composite* $g \circ f$ *of a map* $f : \boldsymbol{N} \to \boldsymbol{R}$ *and a map* $g : \boldsymbol{N} \to \boldsymbol{R}$ *is* ***not*** *defined because the domain of* $g$ *is not the target of* $f$.

2. *If* $f : \boldsymbol{R} \to \boldsymbol{R}$ *and* $g : \boldsymbol{R} \to \boldsymbol{R}$ *are defined by* $f(x) = x^2$ *and* $g(x) = x + 1$ *then*
$$(g \circ f)(x) = x^2 + 1 \text{ and } (f \circ g)(x) = (x + 1)^2.$$

3. *Given* $a \in \boldsymbol{Q}$ *we may write* $a = p/q$ *with* $p \in \boldsymbol{Z}$ *and* $q \in \boldsymbol{N}$*. However,* $p, q$ *are not determined by* $a$*, since if* $a = p/q$ *then also* $a = 2p/2q$*. But we can define maps*
$$f : \boldsymbol{Q} \to \boldsymbol{Z} \quad \text{and} \quad g : \boldsymbol{Q} \to \boldsymbol{N}$$
*by setting* $g(a)$ *to be the least possible* $q$ *and* $f(a)$ *to be the corresponding* $p$*. Let* $h : \boldsymbol{Z} \subset \boldsymbol{Q}$ *be the inclusion. Then*
$$(f \ \circ \ h)(p) = p, \ p \in \boldsymbol{Z}.$$

**Definition 14** *The* ***identity map,*** $id_S$.

*The* ***identity map*** *of a set* $S$ *is the map* $id_S : S \to S$ *defined by*
$$id_S(x) = x, \quad x \in S.$$

**Lemma 2**

*If* $\varphi : S \to T$ *then*
$$id_T \circ \varphi = \varphi = \varphi \circ id_S$$

**Proof:** For $x \in S$,
$$id_T \circ \varphi(x) = id_T(\varphi(x)) = \varphi(x) = \varphi(id_S(x)) = \varphi \circ id_S(x).$$

q.e.d.

**Lemma 3** ***Associativity of composition***

*If* $\varphi : S \to T, \psi : T \to Z,$ *and* $\chi : Z \to W$ *are maps,then*
$$(\chi \circ \psi) \circ \varphi = \chi \circ (\psi \circ \varphi).$$

**Proof:** suppose $x \in S$. Then
$$((\chi \circ \psi) \circ \varphi)(x) = (\chi \circ \psi)(\varphi(x)) = \chi(\psi(\varphi(x))) = \chi \circ (\psi \circ \varphi(x).$$

q.e.d.

## 3.5 Onto, 1-1, and 1-1 correspondences

There are **three very important classes of maps**:

**Definition 15**

1. *A map $\varphi : S \to T$ is* ***onto*** *if $Im\ \varphi = T$. Thus $\varphi$ is onto* ***if and only if*** *every element $y$ can be written in the form $y = \varphi(x)$ for* ***at least one*** *$x \in S$.*

2. *A map $\varphi : S \to T$ is* ***1-1*** *if it maps different elements in $S$ to different elements in $T$. Thus, the map $\varphi$ is 1-1* ***if and only if*** *for all $x_1, x_2 \in S$,*

$$x_1 \neq x_2 \Rightarrow \varphi(x_1) \neq \varphi(x_2).$$

3. *A map $\varphi : S \to T$ is a* ***1-1 correspondence*** *(also called a bijection)* ***if and only if*** *is both onto and 1-1.*

**Caution:** Do **not** confuse the definition of a map with the definition of a 1-1 map:

1. A map $\varphi : S \to T$ associates to each $x \in S$ a **single** element $\varphi(x) \in T$.

2. A map $\varphi : S \to T$ is 1-1 if whenever $x_1$ and $x_2$ are different elements in $S$ then the elements $\varphi(x_1)$ and $\varphi(x_2)$ in $T$ are also different.

In other words:

1. If $\varphi$ is a map from $S$ to $T$, then for each $x \in S$ there is a single $y \in T$ such that $\varphi$ maps $x$ to $y$.

2. If $\varphi$ is a 1-1 map from $S$ to $T$, then

   (a) for each $x \in S$ there is a single $y \in T$ such that $\varphi$ maps $x$ to $y$ (because $\varphi$ is a map)

   **AND**

   (b) for each $y$ in the image of $\varphi$ there is a single $x$ which is mapped to $y$.

**Important Remark:** A map $\varphi : S \to T$ is a 1-1 correspondence if and only if:

1. $T = Im\ \varphi$ ($\varphi$ is onto) **and also**

2. For every $y \in Im\ \varphi$ there is a single $x$ such that $\varphi(x) = y$. ($\varphi$ is $1 - 1$.)

Putting these two statements together we get

$$\varphi \text{ is a 1-1 correspondence}$$

$$\Leftrightarrow \text{ for } \textbf{each } y \in T \text{ there is a } \textbf{single } x \in S \text{ which is mapped to } y.$$

**Example 19** *1. In the third example in Example 17, $\varphi$ is 1-1 but not onto, $\psi$ is neither 1-1 or onto, and $\chi$ is 1-1 but not onto.*

2. *The map $\varphi : \boldsymbol{N} \to \boldsymbol{N}$ given by*

$$\varphi(m) = \begin{cases} 1, & m = 1, 2, \\ m - 1, & m \geq 3 \end{cases}$$

*is onto. It is not 1-1 because both 1 and 2 are mapped to 1.*

3. *The map $\varphi : \boldsymbol{Z} \to \boldsymbol{N}$ given by*

$$\varphi(k) = \begin{cases} -2(k-1), & k \leq 0, \\ 2k - 1, & k \geq 1 \end{cases}$$

*is 1-1 and onto. Thus it is a 1-1 correspondence.*

4. *Let Let $f : \{1, 2, 3, 4\} \to \{a, b, c, d\}$ be defined by $f(1) = c, f(2) = a, f(3) = d,$ and $f(4) = b.$ and let $h : \{a, b, c, d\} \to \{a, b, c, d\}$ be defined by $h(a) = b, h(b) = c, h(c) = d, h(d) = d$. Then*

   (a) *$f \circ h$ is* ***not*** *defined.*

   (b) *$h$ is not onto and is not 1-1.*

   (c) *$h \circ f$ is given by $(h \circ f)(1) = d, (h \circ f)(2) = b, (h \circ f)(3) = d,$ and $(h \circ f)(4) = c$.*

5. *The map $\varphi : \boldsymbol{N} \to \boldsymbol{N}$ defined by*

$$\varphi(n) = n + 1$$

*is 1-1 but not onto, since 1 is not in the image.*

6. *The map $\psi : \boldsymbol{N} \to \boldsymbol{N}$ defined by*

$$\psi(1) = \psi(2) = 1 \textit{ and } \psi(n) = n - 1 \textit{ if } n \geq 3$$

*is onto but not 1-1.*

Now suppose $f : S \to T$ is a 1-1 correspondence. Then for each $y \in T$ there is a unique $x \in S$ such that $f(x) = y$. Thus the rule which associates that $x$ to $y$ is a map from $T$ to $S$,

**Definition 16** *If $f : S \to T$ is a 1-1 correspondence mapping $x$ to $y$, then the map from $T$ to $S$ which maps $y$ to $x$ is called the* ***inverse*** *of $f$ and is denoted $f^{-1}$.*

**Example 20** ***Inverses***

1. *Let* $f : \{1, 2, 3, 4\} \to \{a, b, c, d\}$ *be defined by* $f(1) = c, f(2) = a, f(3) = d$, *and* $f(4) = b$. *Then* $f$ *is a 1-1 correspondence and its inverse is given by* $f^{-1}(a) = 2, f^{-1}(b) = 4, f^{-1}(c) = 1, f^{-1}(d) = 3$.

2. *Let* $A$ *be an* $n \times n$ *matrix, and regard* $A$ *as a map from the set of column vectors to itself by setting* $A(v) = Av$ *(matrix multiplication). If* $B$ *is a second* $n \times n$ *matrix, then* $(A \circ B)(v) = A(B(v)) = A(Bv) = (AB)v$. *Thus the map* $A \circ B$ *is just multiplication by the product matriz* $AB$.

3. *In the example above, suppose* $A$ *is an invertible matrix with inverse matrix* $A^{-1}$. *Then using elementary linear algebra it follows that multiplication by* $A$ *is a bijection, and that the inverse map is multiplication by* $A^{-1}$.

4. *It follows from what you learn in an elementary calculus course that the map* $f : \boldsymbol{R} \to \boldsymbol{R}$ *given by* $f(x) = x^3$ *is a 1-1 correspondence. The inverse maps any real number* $x$ *to its unique cube root.*

5. *Let* $(x_n)_{n\geq 1}$ *be a sequence of real numbers such that for all* $n, x_n < x_{n+1}$. *Then define* $S \subset \boldsymbol{R}$ *by*
$$S = \{x_n \mid n \geq 1\},$$
*and define* $f : \boldsymbol{R} \to \boldsymbol{R}$ *by setting*
$$f(x) = \begin{cases} x, & x \notin S \\ x_{2k-1}, & x = x_{2k}, \\ x_{2k}, & x = x_{2k-1}. \end{cases}$$
*Then* $f$ *is a 1-1 correspondence.*

**Exercise 10** *Maps*

1. *Suppose* $f : S \to T$ *is a map and* $U$ *and* $W$ *are subsets of* $S$.
   (a) *If* $U \cap W = \phi$ *does it follow that* $f(U) \cap f(W) = \phi$*?*
   (b) *If* $f(U) \cap f(W) = \phi$ *does it follow that* $U \cap W = \phi$*?*

2. *List all the maps* $\varphi$ *from* $S = \{1, 2\}$ *to* $T = \{-1, -2\}$ *such that* $Im\ \varphi = T$.

3. *Construct sets* $S, T, U, W$ *and maps* $\varphi : S \to T$ *and* $\psi : U \to W$ *such that a composite* $\psi \circ \varphi : S \to W$ *is not defined but the composite* $\varphi \circ \psi : U \to T$ *is defined.*

4. *If* $f : \boldsymbol{R} \to \boldsymbol{R}$ *and* $g : \{x \in \boldsymbol{R} \mid x \geq 0\} \to \boldsymbol{R}$ *are defined by* $f(x) = x^2$ *and* $g(x) = \sqrt{x}$ *are* $f \circ g$ *and* $g \circ f$ *defined?*

5. *Construct an example of maps* $f : S \to T$ *and* $g : T \to S$ *in which* $f$ *is 1-1 and* $g$ *is onto, but the composite* $g \circ f$ *is neither 1-1 nor onto.*

6. *Construct a 1-1 correspondence from* $\boldsymbol{Z}$ *to* $\boldsymbol{N}$.

7. *If either $S = \phi$ or $T = \phi$ show that $T^S = \phi$.*

**Lemma 4** *1-1 correspondences and inverses*

1. *If $\varphi$ is a 1-1 correspondence then*

$$\varphi^{-1} \circ \varphi = id_S \quad \text{and} \quad \text{and } \varphi \circ \varphi^{-1} = id_T.$$

2. *If there is a map $\psi : T \to S$ satisfying*

$$\psi \circ \varphi = id_S \quad \text{and} \quad \varphi \circ \psi = id_T,$$

*then $\varphi$ is a 1-1 correspondence and $\psi = \varphi^{-1}$.*

**Proof:**

1. For $x \in S$, $x$ is the unique element mapped to $\varphi(x)$ by $\varphi$. Thus by definition

$$\varphi^{-1}(\varphi(x)) = x = id_S(x).$$

Moreover, if $y \in T$ then $y = \varphi(x)$ for a unique $x \in S$. By definition, $x = \varphi^{-1}(y)$. Thus

$$\varphi(\varphi^{-1}(y)) = \varphi(x) = y = id_T(y).$$

2. If $y \in T$ then

$$y = \varphi(\psi(y)) \in Im\ \varphi.$$

Therefore $\varphi$ is onto.

Moreover, if $x_1, x_2 \in S$ and $\varphi(x_1) = \varphi(x_2)$ then

$$x_1 = \psi(\varphi(x_1)) = \psi(\varphi(x_2)) = x_2.$$

Therefore $\varphi$ is 1-1, and so it is a 1-1 correspondence.

Finally,

$$\psi = \psi \circ id_T = \psi \circ (\varphi \circ \varphi^{-1}) = (\psi \circ \varphi) \circ \varphi^{-1} = \varphi^{-1}.$$

q.e.d.

**Proposition 4** *Let $S$ be a non-void set. Then a 1-1 correspondence,*

$$\gamma : P(S) \to \{0, 1\}^S$$

*is given by*

$$\gamma(U)(x) = \begin{cases} 1 & x \in U \\ 0 & x \notin U. \end{cases}$$

**Definition 17** *Characteristic functions*

*The function $\gamma(U)$ is called the* ***characteristic function*** *of U.*

**Proof of Proposition 4:** Define $\chi : \{0,1\}^S \to P(S)$ as follows. If $f \in \{0,1\}^S$ then $f$ is a map from S to {0,1} and we set

$$\chi(f) = \{x \in S \mid f(x) = 1\}.$$

Then for $U \subset S$,

$$(\chi \circ \gamma)(U) = \{x \in S \mid \gamma(U)(x) = 1\} = U.$$

Also, for $f : S \to \{0,1\}$,

$$(\gamma \circ \chi(f)) = \gamma(\{x \in S | \; f(x) = 1\}) = f.$$

Therefore by Lemma 4, $\gamma$ is a 1-1 correspondence with inverse $\chi$. **q.e.d.**

**Theorem 3** *If $S$ is any non-void set then no map from $S$ to $P(S)$ is onto.*

**Proof:** In view of Proposition 4 it is sufficient to prove that no map from $S$ to $\{0,1\}^S$ is onto. We do this by contradiction, and assume for some

$$\varphi : S \to \{0,1\}^S$$

that $\varphi$ is onto.

Now by definition, for each $x \in S$, $\varphi(x)$ is a map from $S$ to $\{0,1\}$. Define

$$f : S \to \{0,1\}$$

by setting

$$f(x) = \begin{cases} 1 & \varphi(x)(x) = 0 \\ 0 & \varphi(x)(x) = 1. \end{cases}$$

We shall show that $f$ is not in the image of $\varphi$ and so $\varphi$ is not onto.

By construction it is never true for any $x \in S$ that $\varphi(x)(x) = f(x)$. But if $\varphi(y) = f$ for some $y \in S$ then by definition,

$$\varphi(y)(z) = f(z), z \in S.$$

In particular,

$$\varphi(y)(y) = f(y),$$

which is impossible. Thus $f$ is not in $Im\varphi$ and so $\varphi$ is not onto.
q.e.d.

**Exercise 11** ***Sets and maps***

1. *Determine which of the following is a map:*

   (a) $\varphi : \boldsymbol{N} \to \boldsymbol{N}$, *defined by*

   $$\varphi(x) = \begin{cases} 1, 2 & x = 1, \\ x + 2 & x > 1. \end{cases}$$

   (b) $\psi : \boldsymbol{R} \to \boldsymbol{R}$, *defined by*

   $$\psi(x) = \begin{cases} 1 & x = 1, \\ x + 2 & x \neq 1. \end{cases}$$

   (c) $\omega : \boldsymbol{C} \to \boldsymbol{C}$ *defined by:*

   $$\omega(x) = \begin{cases} 5 & x = 1, 2, 3 \\ x + 2 & x \neq 1, 2, 3. \end{cases}$$

2. *Which of the maps in Problem 1. are 1-1, onto?*

3. *For $\psi$ and $\omega$ in Problem 1. evaluate $\psi \circ \psi$ amd $\omega \circ \omega$.*

4. *If $S$ and $T$ are finite sets show that $|S \times T| = |S||T|$.*

5. *If $S$ is a finite set show that $|P(S)| = 2^{|S|}$.*

6. *Prove that:*

   (a) *for any set $S$, $id_S$ is a bijection.*

   (b) *The composite of onto set maps $\varphi$ and $\psi$ is onto.*

   (c) *The composite of 1-1 set maps is 1-1.*

   (d) *If $\varphi : S \to T$ and $\psi : T \to W$ are 1-1 correspondences show that the inverse of $\varphi$ and also $\psi \circ \varphi$ are 1-1 correspondences.*

7. *Suppose $\varphi : S \to T$ is a map.*

   (a) *If $T$ is finite and the map is 1-1 show that $S$ is finite and that $|S| \leq |T|$.*

   (b) *If $S$ is finite and the map is onto show that $|T|$ is finite and that $|S| \geq |T|$.*

8. *Suppose $\varphi : S \to T$ is a map and that $S$ and $T$ are finite sets with the same number of elements. Show that the following three conditions are equivalent:*

(a) $\varphi$ *is a 1-1 correspondence.*

(b) $\varphi$ *is onto.*

(c) $\varphi$ *is 1-1.*

9. *Construct two maps* $\alpha$ *and* $\beta$*, both from the same set* $S$ *to itself, such that* $\alpha$ *is 1-1 but not onto and* $\beta$ *is onto but not 1-1.*

10. *If* $S$ *and* $T$ *are finite sets show that there are* $|T|^{|S|}$ *elements in the set of maps from* $S$ *to* $T$*.*

11. *How many elements are there in the set of 1-1 correspondences of a finite set* $S$ *to itself?*

## 3.6 Equivalence Relations

A **relation** between two sets, $S$ and $T$ is a generalization of the idea of a map:

**Definition 18** *A **relation**,* $R$*, between a set* $S$ *and a set* $T$ *is a subset* $R \subset S \times T$ *of the product of* $S$ *with* $T$*.*

*If* $(x, y) \in R$ *we say that* $y$ *is **related** to* $x$ *by the relation* $R$*, and we write* $xRy$*.*

*If* $S$*=*$T$ *we say that* $R$ ***is a relation in*** $S$*.*

**Example 21** *Suppose* $\varphi : S \to T$ *is a map. Then the relation*

$$\{(x, \varphi(x)) \mid x \in S\} \ \subset S \times T$$

*is called the **relation of the map*** $\varphi$*.*

*Thus relations corresponding to set maps are those subsets of* $S \times T$ *that satisfy the following property: **every*** $x \in S$ *is related to a **single*** $y \in T$*.*

*By contrast, **any** subset* $R \subset S \times T$ *is a relation between* $S$ *and* $T$*.*

**Exercise 12** ***Relations***

1. *Identify whether which of the following subsets* $R \subset S \times T$ *is a relation determined by a map:*

   (a) $S$ *is all of* $\boldsymbol{R}$ *and* $T$ *is all of* $\boldsymbol{Z}$*, and* $R = \{(2, -3), (3, -4), (4, -5)\}$.

   (b) $S = \boldsymbol{N}$*,* $T = \boldsymbol{R}$ *and* $R = \{(n, 1/n) | n \in \boldsymbol{N}\}$.

   (c) $S = \boldsymbol{N}$*,* $T = \boldsymbol{N}$ *and* $R = \{(n^2, 1/n) \mid n \in \boldsymbol{N}\}$.

2. *Show that the relation* $\{(x^2, x) \mid x \in \boldsymbol{R}\} \subset \boldsymbol{R}^2$ *is not the relation of a map.*

3. *Which are the integers $n \in \boldsymbol{N}$ such that the relation $\{(x^n, x) \mid x \in \boldsymbol{R}\} \subset \boldsymbol{R}^2$ is the relation of a map from $\boldsymbol{R}$ to $\boldsymbol{R}$?*

4. *If $S$ and $T$ are finite sets, how many relations are there between $S$ and $T$? Compare this with the number of maps from $S$ to $T$ - see Exercise 6.*

Perhaps you have heard the expression,

**You can't see the forest for the trees.**

The speaker is accusing the audience that they are so bound up in the details of the trees that they don't come to grips with the more global properties of the forests. And in fact, sometimes instead of looking at individual trees, we want to look at the forests as individuals. Similarly, instead of considering individual people we might want to talk about cities, which are collections of people. We might want to describe the properties of species, which are collections of animals, rather than the distinctions between individual animals.

This leads to a very important mathematical idea, which we formalize in the following way:

**Definition 19** *A* **partition** *of a set $S$ is a family of non-empty subsets $S_i \subset S$ such that every element $x \in S$ belongs to exactly one subset $S_i$.*

Every partition $\{S_i\}$ of a set $S$ determines a specific relation in $S \times S$; namely, we set $xRy$ if and only if $x$ and $y$ are in the same subset $S_i$. This is called the **relation of the partition.**

**Example 22**

1. *The single set $S$ is a partition of a non-empty set $S$. The corresponding relation is $xRy$ for every $x, y \in S$. Thus $R = S \times S$.*

2. *The family of subsets $S_x, x \in S$ of a non-empty set $S$ is a partition of $S$, since every $x$ belongs to a single $S_x$. The corresponding relation iis $xRy$ if and only if $x = y$.*

3. *A partition of the set, $S$, of sophomore students at the University of Maryland is defined by: For each possible GPA, $\alpha$, $S_\alpha$ is the set of students whose GPA is $\alpha$.*

**Lemma 5** *The relation of a partition $\{S_i\}$ of a set $S$ satisfies the following properties:*

1. *For every $x \in S$, $xRx$. (The relation is* ***reflexive****.)*

2. *If $xRy$ then also $yRx$. (The relation is* ***symmetric****.)*

3. *If $xRy$ and $yRz$ then $xRz$. (The relation is* ***transitive****.)*

**Proof:**:

1. The relation is reflexive because $x$ and $x$ belong to the same $S_i$ .

2. If $xRy$ then by definition $x$ and $y$ are in the same $S_i$ and so $yRx$.
   Therefore the relation is symmetric.

3. We have to show that if $xRy$ and $yRz$ then $xRz$.

   By hypothesis, $x$ belongs to exactly one $S_i$.
   Since $xRy, y$ must also be in that $S_i$.
   Since $yRz, z$ must also be in that $S_i$.
   Thus $y$ and $z$ are in the same $S_i$.
   Therefore by definition, $xRz$ and the relation is transitive.

**q.e.d.**

**Definition 20** *An* ***equivalence relation*** *in a set $S$ is a reflexive, symmetric, and transitive relation. Equivalence relations are denoted by $x \sim y$.*

Lemma 5 states that the relation of a partition is an equivalence relation. Conversely we have

**Proposition 5** *Every equivalence relation, $\sim$, in a set $S$ is the relation of a unique partition of $S$.*

**Proof:** Fix an equivalence relation, $\sim$, in $S$.

For each $x \in S$ define a subset $S(x) \subset S$ by

$$S(x) = \{y \in S \mid y \sim x\}.$$

Now let $\{S_i\}$ denote the family of distinct subsets of $S$ such that each $S_i = S(x)$ for some $x \in S$.

We will prove three things:

1. This is a partition of $S$.

2. $\sim$ is the relation of the partition.

3. This is the only partition with $\sim$ as its relation.

**Step 1**. This is a partition of $S$.

By reflexivity each $x \in S(x)$ and so the $S_i$ are not empty and every $x$ in $S$ belongs to some $S_i$.

Thus to show this is a partition we have to show that any element of $S$ can belong to only one subset $S_i$.

In other words we have to show that if $y \in S(x)$ and $y \in S(z)$ then $S(x) = S(z)$.

We first prove that for any $x \in S$, if $u \in S(x)$ then

$$S(u) \subset S(x).$$

In fact, since $u \in S(x)$ we have $u \sim x$, and if $v \in S(u)$ we have $v \sim u$.
Thus $v \sim u \sim x$, and since the relation is transitive, $v \sim x$.
It follows that $v \in S(x)$; i.e. $S(u) \subset S(x)$.

Next we prove that if $u \in S(x)$ then

$$S(x) \subset S(u).$$

In fact, since $u \sim x$ and the relation is symmetric, $x \sim u$.
Therefore $x \in S(u)$ and by what we have just shown it follows that

$$S(x) \subset S(u).$$

Altogether we have proved that if $u \in S(x)$ then $S(x) = S(u)$.

Finally, if $y \in S(x)$ and $y \in S(z)$, then by the equality we just proved

$$S(y) = S(x) = S(z).$$

Therefore $\{S_i\}$ is a partition and Step 1 is proved.

**Step 2**. The equivalence relation $\sim$ is the relation of the partition.

Denote the relation of the partition by R, so that $xRy$ if and only if $x$ and $y$ are in the same subset $S_i$ of the partition.

Then by definition,

$$xRy \Leftrightarrow x, y \in S(w), \text{ some } w \in S.$$

But we know that if $x, y \in S(w)$ then $S(x) = S(w) = S(y)$ and so $x \sim y$. Thus $R = \sim$.

**Step 3**. The partition $\{S_i\}$ is the unique partition which has $\sim$ as its equivalence relation.

Let $\{T_\alpha\}$ be a partition of $S$ with $\sim$ as its equivalence relation.

If $x \in S$ then the subset $T_\alpha$ in the partition containing $x$ will satisfy

$$y \sim x \text{ if and only if } y \in T_\alpha.$$

Thus $T_\alpha = S(x)$.
It follows that every $T_\alpha$ is one of the $S(x)$.
Since every element of $S$ is in some $T_\alpha$ it follows that every $S(x)$ is one of the $T_\alpha$.
Therefore the partition $\{T_\alpha\}$ is the partition $\{S_i\}$. **q.e.d.**

**Definition 21** *If $\sim$ is an equivalence relation in a set $S$ then the elements $S_i$ of the corresponding partition are called the* ***equivalence classes*** *of the relation.*

**Exercise 13** ***Equivalence relations***

1. *Provide a complete proof for the statement in Example 6.3.*
2. *Two UM students are related if they both take at least one class in common. Is this an equivalence relation?*
3. *Two UM students are related if they take all their classes in common. Is this an equivalence relation?*
4. *Suppose $f \colon S \to T$ is a map between two sets. Say $x$ and $y$ in $S$ are related if $f(x) = f(y)$. Is this an equivalence relation?*
5. *Suppose $S = \cup_{i=1}^m S_i$ is a partition of a set with $n$ elements for some natural number $n$.*

   (a) *If each $S_i$ has the same number of elements, $k$,show that $k$ divides $n$ and that $n/k = m$.*

   (b) *If each $S_i$ has $i$ elements find $n$.*

   (c) *If each $S_i$ has $i^3$ elements, find $n$.*

   (d) *If each $S_i$ has an odd number of elements show that $m$ is even if and only if $n$ is even.*

6. *Recall that $P(\boldsymbol{Q})$ denotes the set of all subsets of $\boldsymbol{Q}$. Thus the* ***elements*** *of $P(\boldsymbol{Q})$ are the* ***subsets*** *$S \subset \boldsymbol{Q}$. Define a relation*

$$\alpha \subset P(\boldsymbol{Q}) \times P(\boldsymbol{Q})$$

   *by setting $S\alpha T$ if*

   (a) *for every $a \in S$ there is some $b \in T$ such that $b \geq a$,* ***and*** *(b) for every $c \in T$ there is some $d \in S$ such that $d \geq c$.*

*Then*

*(a) Show that $\alpha$ is an equivalence relation.*

*(b) Show that if $\{S_i\}$ is an equivalence class of subsets then*

$$\bigcup_i S_i$$

*is an element in that equivalence class.*

*(c) Show that if $S \subset \boldsymbol{Q}$ is not equivalent to $\boldsymbol{Q}$ then for some rational number, $a$, $a > x$ for all $x \in S$.*

*(d) If $a \in \boldsymbol{Q}$ are the sets*

$$S = \{b \in \boldsymbol{Q} \mid b < a\} \quad \textit{and} \quad T = \{b \in \boldsymbol{Q} \mid b \leq a\}$$

*equivalent?*

7. *Let $T^S$ denote the set of maps from a set $S$ to a set $T$. Define a relation $R \subset T^S \times T^S$ by setting*

$$fRg$$

*if $f(x) = g(x)$ except for finitely many points $x \in S$ (depending, of course, on $f$ and $g$). Show this is an equivalence relation.*

# Chapter 4

# The Real Numbers

The bedrock of analysis is an understanding of the real numbers. Notice the difference between the rational numbers and the real numbers: we can describe the rational numbers explicitly as quotients of one integer by another. No such simple expression is available for the reals. We may have an intuitive understanding of these as the points on the real number line, as distances, or as (possibly) infinite decimals, but we cannot use "intuitive understandings" to make rigorous proofs.

We solve this problem by showing that a real number can be approximated arbitrarily well by a rational number. Explicitly:

**If $x \in \mathbf{R}$ then for any $\delta > 0$ there is a rational number $a \in \mathbf{Q}$**

**such that**

$$|x - a| < \delta.$$

This allows us to extend properties of the rationals to the reals in a rigorous way.

The concept of "**approximation arbitrarily well**" is very important throughout the rest of this course. Thus, more generally, if $S \subset \mathbf{R}$ is any subset of the reals we say that $x \in \mathbf{R}$ can be **approximated arbitrarily well by numbers in** $S$ if for every $\delta > 0$ there is a number $y \in S$ such that $|x - y| < \delta$.

Our first step is to recall the basic properties of the rationals.

## 4.1 Properties of the Rational Numbers

Here are the two fundamental concepts for the rationals with which we are familiar:

1. **Algebra:** The operations of addition, subtraction, multiplication and division.

2. **Order:** If $b$ and $a$ are rational numbers then

$$a < b \Leftrightarrow b - a = m/n \text{ for some } m, n \in \mathbf{N}.$$

**Notation:** We use $b > a$ to mean the same thing as $a < b$.

The basic properties of the ordering in the rationals are contained in the next Lemma. While they are utterly and totally familiar we shall give formal proofs to provide more examples of what a proof looks like.

**Lemma 6**

1. *For each rational number $a \in \boldsymbol{Q}$ exactly one of the following three possibilities is true:*

   *(a)* $a > 0$

   *(b)* $a = 0$

   *(c)* $a < 0$

   *In particular, if $a, b \in \boldsymbol{Q}$ then*

$$b > a \Leftrightarrow b - a > 0.$$

2. ***If*** *$a, b, c$ are any three rational numbers,* ***then****:*

   *(a)* ***If*** *$a < b$ and $b < c$* ***then*** *$a < c$.*

   *(b)* *$a + b < a + c$* ***if and only if*** *$b < c$.*

   *(c) Multiplication by a positive preserves inequalities.*

   *(d) Multiplication by a negative reverses inequalities.*

3. ***If*** *$a \in \boldsymbol{Q}$* ***then*** *$a > 0 \Leftrightarrow a > 1/n$ for some $n \in \boldsymbol{N}$.*

**Proof:**: We prove each statement separately.

1. By definition $a = m/n$ with $m \in \mathbf{Z}$ and $n \in \mathbf{N}$. Thus exactly one of the following three possibilities is true:

   (a) $m \in \mathbf{N}$,

   (b) $m = 0$, or

   (c) $-m \in \mathbf{N}$.

   In the first case $a > 0$.
   In the second case, $a = 0$.
   In the third case $a < 0$.

2. (a) If $a < b$ and $b < c$ then

$$b - a = p/q \textit{ and } c - b = m/n,$$

with $p, q, m, n \in \mathbf{N}$.

It follows that

$$c - a = (c - b) + (b - a) = m/n + p/q > 0.$$

(b) Since $(a + c) - (a + b) = c - b$ it follows that

$$(a + c) - (a + b) > 0 \Leftrightarrow c - b > 0.$$

(c) If $c > 0$ and $b - a > 0$, then for some $p, q, m, n \in \mathbf{N}$

$$c = p/q \textit{ and } b - a = m/n\ .$$

Therefore

$$cb - ca = c(b - a) = pm/qn > 0.$$

(d) If $c < 0$ and $a < b$ then $-c > 0$ and,

$$-c(b - a) > 0.$$

Adding $c(b - a)$ to both sides gives $0 > c(b - a)$ and so

$$ca > cb$$

3. If $a > 0$,we have for some $m, n \in \mathbf{N}$ that

$$a = m/n > 1/n.$$

Conversely, for any $n \in \mathbf{N}$, $1/n > 0$.
Thus if $a > 1/n$ then $a > 1/n > 0$.

**q.e.d.**

## 4.2 Introducing the Real Numbers and their Inequalities

Recall that while we may have an intuitive understanding of the real numbers as the points on the x-axis of the plane, or as distances, or as (possibly) infinite decimals, we cannot use "intuitive understandings" to make rigorous proofs. Thus we introduce the reals by listing four elementary properties as axioms, from which all the other properties will be deduced. To rigorously extend properties of the rationals to the reals we sall use the fact that any real number can be aproximated by rational numbers with arbirarily small error. Thus working with inequalities will be an important technique for the rest of this course.

Of the four basic properties we shall assume about the real numbers, the first three are immediately below. The fourth will be stated in Se. 4.5.

**Property One:** The real numbers are a set, denoted by $\mathbf{R}$, and containing the rational numbers.

**Property Two (Algebra):** The operations of addition, subtraction, multiplication and division are defined for real numbers, coinciding with the old operations in the rationals, and with the same properties.

**Property Three (Order):** The ordering of the rationals extends to an ordering of the real numbers such that

1. For each real number $x$ exactly one of the following three possibilities is true:
   (a) $x > 0$, in which case we say $x$ is **positive**.
   (b) $x = 0$.
   (c) $x < 0$, in which case we say $x$ is **negative.**
2. $x < y \Leftrightarrow y - x > 0$.
3. If $x < y$ and $y < z$ then $x < z$.
4. The product of positive real numbers is positive.
5. If $x \in \mathbf{R}$ then there is a natural number $m$ such that

$$x < m.$$

**Notation:** We shall write $y > x$ to mean $x < y$.

**Lemma 7**

1. *If $x, y, z$ are any three real numbers, then:*
   *(a)* ***Addition preserves inequalities:*** $x + y < x + z \Leftrightarrow y < z$.
   *(b) Multiplication by a* ***positive*** *real number* ***preserves inequalities****:*

$$z > 0 \quad \textit{and} \quad x < y \Rightarrow zx < zy.$$

   *(c) Multiplication by a* ***negative*** *real number* ***reverses inequalities****:*

$$z < 0 \quad \textit{and} \quad x < y \Rightarrow zx > zy.$$

(d) *If* $x > 0$ *then* $1/x > 0$.

(e) *If* $0 < x < y$ *then* $1/y < 1/x$.

2. *For each* $x \in \boldsymbol{R}$ *there is a natural number* $n$ *such that*

$$-n < x.$$

3. *For each positive real number* $x$ *there are natural numbers* $k, m$ *such that*

$$1/k < x < m.$$

**Proof:**: We prove each statement separately.

1. Suppose $x, y, z \in \mathbf{R}$.

(a) Since $(x + z) - (x + y) = z - y$ it follows from Property Three that

$$x + y < x + z \Leftrightarrow y < z.$$

(b) If $0 < x$ then

$$-x = -x + 0 < -x + x = 0.$$

Conversely, if $-x < 0$, then adding $x$ to both sides gives $0 < x$.

(c) Since $x < y$ then $y - x > 0$.
Since $z > 0$ and the product of positive real numbers is positive (Property 3), it follows that

$$zy - zx = z(y - x) > 0.$$

(d) If $z < 0$ then $z < z - z$ and subtracting $z$ from both sides gives $0 < -z$. Therefore if $x < y$ then

$$0 < (-z)(y - x).$$

Adding $z(y - x)$ to both sides gives $z(y - x) < 0$.

(e) We show by contradiction that

$$x > 0 \Rightarrow 1/x > 0.$$

In fact, suppose $1/x \leq 0$. Then by what we just proved, $-1/x \geq 0$. Since the product of positives is positive (Property 3) it would follow that

$$-1 = (-1/x)x \geq 0,$$

which is false. It follows by contradiction that $1/x > 0$.

(f) Since $x, y > 0$, $1/x, 1/y > 0$. Therefore

$$1/xy = (1/x)(1/y) > o.$$

Multiplication by $1/xy$ therefore preserves the inequality $x < y$ and this gives $1/y < 1/x$.

2. By Property 3 applied to $-x$, for some $n \in \mathbf{N}$ we have $-x < n$. Since multiplication by $-1$ reverses inequalities it follows that

$$-n < x.$$

3. Since $x > 0$, $1/x > 0$. Thus by Property Three there are natural numbers $k, m \in \mathbf{N}$ such that

$$1/x < k \quad \text{and} \quad x < m.$$

By what we proved above this implies that $1/k < x$ and so

$$1/k < x < m.$$

**q.e.d.**

The preceding Lemma establishes properties you have been taking for granted for years. Henceforth we will use these properties without referring to the Lemma for justification.

The next result is **absolutely fundamental** for this course. It states that **any real number can be approximated arbitrarily well by rational numbers**.

**Theorem 4** ***(Sandwich theorem)*** *Suppose $x \in \boldsymbol{R}$ and $k \in \boldsymbol{N}$.*

1. *Then there are rational numbers $a, b$ such that*

$$a < x < b \quad \textit{and} \quad b - a < 1/k.$$

2. *In particular,*

$$0 < x - a < 1/k \quad \textit{and} \quad 0 < b - x < 1/k.$$

**Proof:**

1. By Property Three amd Lemma 7 there are natural numbers $n, m$ such that

$$-n < x < m.$$

Set $p = m(2k+1)$ and $q = n(2k+1)$. Then we may rewrite this inequality as

$$\frac{-q}{2k+1} < x < \frac{p}{2k+1}.$$

Observe first that if $r \in \mathbf{Z}$ and $x < r/(2k+1)$ then

$$\frac{-q}{2k+1} < x < \frac{r}{2k+1},$$

and so $-q < r$.

Therefore there is a least integer $r$ such that

$$x < \frac{r}{2k+1}.$$

Then

$$x \geq \frac{r-1}{2k+1} \quad \text{and so} \quad x > \frac{r-2}{2k+1}.$$

Thus

$$\frac{r-2}{2k+1} < x < \frac{r}{2k+1}$$

and

$$\frac{r}{2k+1} - \frac{r-2}{2k+1} = \frac{2}{2k+1} < 1/k.$$

Therefore the Theorem is satisfied with

$$a = \frac{r-2}{2k+1} \quad \text{and} \quad b = \frac{r}{2k+1}.$$

2. This is immediate from what was just proved.

**q.e.d.**

**Proposition 6** *If $x < y$ are real numbers then for some $b \in \boldsymbol{Q}$,*

$$x < b < y.$$

**Proof:** By Lemma 7, for some $k \in \mathbf{N}, 1/k < y - x$. and so $x + 1/k < y$. Now by the Sandwich theorem, for some $b \in \mathbf{Q}$,

$$x < b \quad \text{and} \quad b - x < 1/k.$$

Since $b$ was chosen so that $x + 1/k < y$ it follows that

$$b < x + 1/k < y.$$

**q.e.d.**

**Proposition 7** *If $x > 0$ and $y > 1$ are real numbers then for some $p \in \boldsymbol{N}$,*

$$1/y^p < x.$$

**Proof:** Since $y - 1 > 0$, by Lemma 7, for some $k \in \mathbf{N}, y - 1 > 1/k$. Therefore

$$y > (1/k) + 1.$$

Now apply the Binomial theorem to obtain that for **any** $p \in \mathbf{N}$:

$$((1/k) + 1)^p = 1 + p/k + \Sigma_{i=2}^{p} \left( \begin{array}{c} p \\ i \end{array} \right) (1/k)^i > p/k.$$

Moreover, (by the Difference theorem),

$$y^p - (1/k + 1)^p = (y - (1/k + 1)\Sigma_{i=0}^{p-1} y^i (1/k + 1)^{p-1-i} > 0.$$

Therefore

$$y^p > (1/k + 1))^p > p/k.$$

It follows from Lemma 7 that for any $p \in \mathbf{N}$

$$1/y^p < k/p.$$

Finally, by Lemma 7, for some $n \in \mathbf{N}$, $x > 1/n$ . Choose $p = kn$. Then

$$1/y^p < k/kn = 1/n < x.$$

q.e.d.

**Exercise 14** ***Inequalities***

1. *Is it true that there are rational numbers $a < b$ such that for any natural number $k, b - a < 1/k$?*
2. *If $x, y, z, w$ are positive real numbers such that $x < y$ and $z < w$ show that $xz < yw$. State the converse and either prove it or provide a counterexample to show it is false.*
3. *Show that if $x < y$ are real numbers then there are infinitely many rational numbers $b$ such that $x < b < y$.*
4. *Suppose $x, y, z$ are any real numbers. Show that if $x > y + z$ then there are rational numbers $a > y$ and $b > z$ such that*

$$x > a + b.$$

5. *Show that the product of a positive real number with a negative real number is negative.*
6. *Let $x$ and $y$ be positive real numbers. Show that if $x > 1$ then (i) $xy > y$, (ii) $0 < 1/x < 1$, and (iii) $0 < y/x < y$.*
7. *If $a < b$ are non-negative real numbers show that for any $p \in \boldsymbol{N}$, $b^p - a^p \leq p(b - a)b^{p-1}$. (Hint:use the Difference theorem).*

## 4.3 Absolute value

Recall that the ***absolute value*** of a real number $x$ is denoted by $|x|$ and is defined by:

$$|x| = x \text{ if } x \geq 0 \text{ and } |x| = -x \text{ if } x < 0.$$

**Note**:

1. $-x$ is positive when $x$ is negative!

2. For every $x \in \mathbf{R}$:

$$|x| \geq 0$$

   and

$$|x| = 0 \Leftrightarrow x = 0.$$

3. $|x|$ is the larger of $x$ and $-x$.

4. Thus the statement $|x| < \varepsilon$ is the same as $x < \varepsilon$ *and* $-x < \varepsilon$, and hence

$$|x| < \varepsilon \Leftrightarrow -\varepsilon < x < \varepsilon.$$

The next Proposition illustrates the difference between algebra and analysis. In algebra we show equality directly. In analysis we sometimes show two numbers are the same by showing that their difference is arbitrarily small!

**Proposition 8** *If $x, y$ are real numbers such that*

$$|x - y| < 1/n \quad \textit{for every} \quad n \in \boldsymbol{N},$$

*then $x = y$.*

**Proof:**: We prove this by contradiction.
Suppose the conclusion $x = y$ is false.
Then since $x \neq y$, it follows that $x - y \neq 0$.
Therefore $|x - y| > 0$.
Now by Property 3, $|x - y| > 1/n$ for some $n \in \mathbf{N}$.
This contradicts the statement $|x - y| < 1/n$ for **every** $n \in \mathbf{N}$.
**q.e.d.**

**Lemma 8** *For any real numbers $x$ and $y$:*

1. $|xy| = |x||y|$.

2. $x^2 = |x^2|$ *and so* $x^2 \geq 0$.

3. $|x + y| \leq |x| + |y|$ *(The* ***triangle inequality****).*

*4.* $||x| - |y|| \le |x - y|$.

**Proof:**: We prove each statement separately.

1. $|xy|$ and $|x||y|$ are both non-negative numbers equal either to $xy$ or to $-xy$.
   Therefore in this case they are equal.

2. Either $|x| = x$ or $|x| = -x$. In either case $x^2 = |x^2| \ge 0$.

3. This is obvious if $x = 0 = y$. Thus we may suppose one of $|x|, |y|$ is positive. Now

   $$|x+y|^2 = (x+y)^2 = x^2+2xy+y^2 = |x|^2+2xy+|y|^2 \le |x|^2+2|x||y|+|y|^2 = (|x|+|y|)^2.$$

   Therefore $|x + y|^2 \le (|x| + |y|)^2$. It follows that

   $$0 \le (|x| + |y|)^2 - |x + y|^2 = [|x| + |y| - |x + y|] \cdot [(|x| + |y|) + |x + y|].$$

   Since the factor on the right is positive it follows that

   $$|x| + |y| - |x + y| \ge 0.$$

4. By the triangle inequality,

   $$|x - y| + |y| \ge |x - y + y| = |x|,$$

   and

   $$|x - y| + |-x| \ge |x - y - x| = |y|.$$

   Therefore

   $$|x - y| \ge |x| - |y| \quad \text{and} \quad |x - y| \ge |y| - |-x| = |y| - |x|$$

   Therefore

   $$|x - y| \ge ||x| - |y||.$$

**q.e.d.**

The next Proposition is **fundamental** for this course.

**Proposition 9** *Suppose $a \in \boldsymbol{R}$ and $\varepsilon \in \boldsymbol{R}$ is positive. Then for $x \in \boldsymbol{R}$*

$$|x - a| < \varepsilon \Leftrightarrow a - \varepsilon < x < a + \varepsilon.$$

**Proof:** The statement $|x-a|<\varepsilon$ is the same as

$$x-a<\varepsilon \text{ and } a-x<\varepsilon.$$

But

$$x-a<\varepsilon \Leftrightarrow x<a+\varepsilon \text{ and } a-x<\varepsilon \Leftrightarrow x>a-\varepsilon.$$

Thus

$$|x-a|<\varepsilon \Leftrightarrow a-\varepsilon<x<a+\varepsilon.$$

**q.e.d.**

**Exercise 15** ***Absolute value***

1. *If $x$ is a positive real number show that for some $\varepsilon>0$,*

$$y\in \boldsymbol{R} \quad \text{and} \quad |x-y|<\varepsilon \Rightarrow y>0.$$

.

2. *If $x,z\in \boldsymbol{R}$ show that for each $\varepsilon>0$ there is some $\delta>0$ such that if $y\in \boldsymbol{R}$ satisfies $|y-x|<\delta$ then $|zy-zx|<\varepsilon$.*

3. *If $x\in \boldsymbol{R}$ show that for each $\varepsilon>0$ there is some $\delta>0$ such that if $y\in \boldsymbol{R}$ satisfies $|y-x|<\delta$ then $|y^2-x^2|<\varepsilon$. (Hint: Use $y^2-x^2=(y-x)(y+x)$.)*

4. *If $x\in \boldsymbol{R}$ and $x\neq 0$ show that for each $\varepsilon>0$ there is some $\delta>0$ such that if $y\in \boldsymbol{R}$ satisfies $|y-x|<\delta$ then $y\neq 0$ and $|1/y-1/x|<\varepsilon$.*

5. *If $p,q$ are natural numbers and $\varepsilon$ is a positive real number, show thst for some natutal number $N$*

$$n\geq N \quad \text{and} \quad n\in \boldsymbol{N} \Rightarrow |p/n-q/n|<\varepsilon.$$

## 4.4 Bounds

**Definition 22** ***Bounds***

1. *A nonvoid subset $S\subset \boldsymbol{R}$ is* **bounded above** *if for some $b\in \boldsymbol{R}$,*

$$x\leq b \quad \text{for all} \quad x\in S.$$

*In this case we write*

$$S\leq b \quad \text{or} \quad b\geq S,$$

*and say $b$ is an* ***upper bound*** *for $S$.*

2. *A nonvoid subset $S\subset \boldsymbol{R}$ is* **bounded below** *if for some $a\in \boldsymbol{R}$,*

$$a\leq x \quad \text{for all} \quad x\in S.$$

*In this case we write*

$$a\leq S \quad \text{or} \quad S\geq a,$$

*and say $a$ is a* ***lower bound*** *for $S$.*

3. *A subset $S \subset \boldsymbol{R}$ is* **bounded** *if it is both bounded above and bounded below.*

**Lemma 9** *Suppose $S$ and $T$ are non-void subsets of $\boldsymbol{R}$. Then every element of $S$ is a lower bound for $T$ if and only if every element of $T$ is an upper bound for $S$.*

*In this case we write*

$$S \leq T \quad \text{or} \quad T \geq S.$$

**Proof:**: The statement that every element of $S$ is a lower bound for $T$ is true if and only if $x \leq T$ for every $x \in S$.
This is equivalent to the statement:

$$x \leq y \ \text{ for all } \ x \in S \ \text{ and all } \ y \in T.$$

But this is equivalent to saying that every $y \in T$ is an upper bound for $S$.
**q.e.d.**

**Example 23** ***Bounds***

1. *Any real number $x \leq 1$ is a lower bound for $\boldsymbol{N}$.*

2. *$\boldsymbol{N}$ is not bounded above.*

   *In fact if $x$ is any real number then by Property 3 there is some natural number $m$ such that $x < m$.*
   *Thus $x$ is not an upper bound for $\boldsymbol{N}$, and so $\boldsymbol{N}$ does not have an upper bound.*

3. *$S = \{1/n \mid n \in \boldsymbol{N}\}$ is bounded above by any number $\geq 1$ and bounded below by any number $\leq 0$.*

4. *The rationals are not bounded below or above.*

5. *The set $S$ of real numbers whose squares are less than 2 is bounded above by 2 and below by $-2$.*

   *In fact, if $x > 2$ then $x^2 = xx > 2x > 4$ and so $x \notin S$.*
   *Also, if $x < -2$ then $-x > 2$ and $x^2 = xx = (-x)(-x) > 4$ and again, $x \notin S$.*
   *Thus*

$$-2 \leq S \leq 2.$$

**Exercise 16** ***Bounds***

1. *Show that if $x$ is a positive real number then $x + 1$ is not a lower bound for $\boldsymbol{N}$.*

2. *Show that if $b$ is an upper bound for a non-void set $S \subset \boldsymbol{R}$ then every $c > b$ is also an upper bound for $S$.*

3. *Give an example of a non-void bounded set $S \subset \boldsymbol{R}$ which contains an upper bound but does not contain a lower bound.*

4. *Show that a non-void set $S \subset \boldsymbol{R}$ cannot contain two different upper bounds.*

5. *Show that the set of real numbers of the form $x/y$ with $|x| > |y| > 0$ is not bounded above or below.*

6. *Show that the set of real numbers of the form $x/y$ with $|y| > |x| > 0$ is bounded. Find explicit upper and lower bounds and prove your answers are correct.*

7. *Suppose $S \subset T \subset \boldsymbol{R}$ are non-void sets. If $S \leq T$ show that $S$ has exactly one element.*

8. *More generally, suppose $S \subset \boldsymbol{R}$ and $T \subset \boldsymbol{R}$ are non-void sets. If $S \leq T$ show that $S \cap T$ can have at most one element. If $S \cap T$ is non void show that its unique element is an upper bound for $S$ and a lower bound for $T$.*

## 4.5 Least Upper and Greatest Lower Bounds

In everything we have done so far **there is nothing to suggest that there are real numbers which are not rationals!**

All we have assumed about the real numbers at this point is listed in Properties One, Two, and Three at the start of Section 4.2.

We correct this with one final property, still as an axiom. This is the **fundamental property of the real numbers** and is the key fact which makes analysis possible.

**Property Four:** Let $S$ be a non empty subset of $\mathbf{R}$ which is bounded above. Then $S$ has an upper bound, $b$ with the following property:

**If** $y$ is any upper bound for $S$ **then** $y \geq b$.

**Lemma 10** *The upper bound $b$ in Property Four is unique.*

**Proof:**: Suppose $c$ satisfies the same condition as $b$.
Then $c$ is an upper bound for $S$.
Therefore $c \geq b$.
But $b$ is an upper bound for $S$.
Therefore $b \geq c$.
Therefore $b = c$.
**q.e.d.**

**Definition 23** *The real number $b$ in Property Four is called the* ***least upper bound*** *for $S$ and is denoted by* ***lub(S)*** *or by* ***sup(S)****.*

**Important Remark:** If $S$ is bounded above, **sometimes** $lub(S)$ **is in** $S$ **and sometimes it is not**!

For example if

$$S = \{x \in \mathbf{R} | \ x \leq 1\}$$

or if

$$S = \{x \in \mathbf{R} | \ x < 1\},$$

then in both cases $lub(S) = 1$. However, in the first case, $1 \in S$ and in the second case $1 \notin S$.

Analogous to least upper bounds, a non empty set which is bounded below has a **greatest lower bound**:

**Proposition 10** *Suppose $S$ is a non empty subset of* ***R*** *which is bounded below. Then there is a unique lower bound, a with the following property:*

*__If__ $y$ is any lower bound for $S$ __then__ $y \leq a$.*

**Definition 24** *The real number $a$ in Proposition 10 is called the* ***greatest lower bound*** *for $S$ and is denoted by* ***glb(S)*** *or by* ***inf(S)****.*

**Proof of Proposition 10:** Let $-S = \{-x| \ x \in S\}$.
If $y$ is a lower bound for $S$ then $y \leq S$.
Since multiplication by $-1$ reverses inequalities, $-y \geq -S$.
Therefore $-S$ is bounded above, and so by Property Four, it has a least upper bound, $b$.
Set $a = -b$.
Since $b \geq -S$, it follows that $a = -b \leq S$.

Now let $y$ be any lower bound for $S$. Then $y \leq S$ and so

$$-y \geq -S.$$

Thus $-y$ is an upper bound for $-S$ and so $-y \geq b$.
Therefore $y \leq -b = a$ and so $a = glb(S)$. Its uniqueness is proved in the exact same way as the uniqueness of lub's.
**q.e.d.**

We recap what it means for $b$ to be the least upper bound of a non-void set $S$. It means exactly this:

$$S \leq b \ \ \textbf{and} \ \ \text{if } u < b \text{ then } u \text{ is not an upper bound for } S$$

A useful way of expressing this is given in the next Proposition:

**Proposition 11** *Suppose $S$ is a non-void subset of $\boldsymbol{R}$. Then*

1. *$b$ is the least upper bound of $S$ if and only if*

   (a) *$S \leq b$, and*

   (b) *for all $\varepsilon > 0$ there is an $x \in S$ such that $b - x < \varepsilon$.*

2. *$a$ is the greatest lower bound of $S$ if and only if*

   (a) *$S \geq a$, and*

   (b) *for all $\varepsilon > 0$ there is an $x \in S$ such that $x - a < \varepsilon$.*

**Proof:** For the first assertion, suppose first that (a) and (b) are satisfied.Then $b$ is an upper bound. To show that $b = lub(S)$ we need to show that If $u < b$ then $u$ is **not** an upper bound for $S$.

But if $u < b$, then $b - u > 0$. Therefore by hypothesis (b), for some $x \in S, b - x < b - u$.
Therefore $x > u$ and $u$ is not an upper bound for $S$.

Conversely, suppose $b$ is the least upper bound for $S$. Then $b \geq S$.
Moreover, if $\varepsilon > 0$, then $b - \varepsilon < b$ and so $b - \varepsilon$ is not an upper bound for $S$.
Therefore, for some $x \in S$

$$b - \varepsilon < x.$$

The second assertion is proved in the same way.
**q.e.d.**

**Example 24**

1. *$glb(\boldsymbol{N}) = 1$.*

   *In fact, since $1 \in \boldsymbol{N}$, any lower bound $x$ for $\boldsymbol{N}$ satisfies $x \leq 1$*
   *But $1 \leq n$ for all $n \in \boldsymbol{N}$. Thus 1 is a lower bound for $\boldsymbol{N}$ and so it is the greatest lower bound: $glb(\boldsymbol{N}) = 1$.*

2. *Let $S = \{(x+1)/x \mid x \in \boldsymbol{R}$ and $x \geq 1\}$. Then $1 = glb(S)$.*

   *In fact for $x \geq 1$*

   $$1 \leq 1 + 1/x.$$

   *Moreover, if $y > 1$ then by Property 3 there is some $n \in \boldsymbol{N}$ with $1/n < y - 1$. Thus*

   $$1 + 1/n < y$$

   *and so $y$ is not a lower bound for $S$. Thus 1 is the greatest lower bound.*

**Exercise 17** ***glb and lub***

1. *If $b \in S$ is an upper bound for a non-void set $S \subset \boldsymbol{R}$ show that it is the least upper bound.*

2. *Show that $0$ is the greatest lower bound for $\{1/n \mid n \in \boldsymbol{N}\}$.*

3. *Let $a \in \boldsymbol{R}$ and let $S = \{x \in \boldsymbol{R} \mid |x - a| < 1/2\}$. Show that $S$ is bounded. Then find $lub(S)$ and $glb(S)$ and prove your answers are correct.*

4. *Let $S$ be a a non-void subset of $\boldsymbol{R}$ which is bounded above. Show that the following conditions are equivalent on an upper bound, $b$, for $S$:*

   (a) *$b = lub(S)$.*

   (b) *For each $\varepsilon > 0$ there is an element $x \in S$ such that $b - \varepsilon < x \leq b$.*

   (c) *For each $\varepsilon > 0$ there is an element $x \in S$ such that $0 \leq b - x < \varepsilon$.*

5. *Suppose $S \subset T$ are subsets of $\boldsymbol{R}$:*

   (a) *If $T \subset \boldsymbol{R}$ is a non-void set bounded below show that $S$ is bounded below and that $glb(T) \leq glb(S)$;*

   (b) *If $T$ is bounded above show that $S$ is bounded above and that $lub(T) \geq lub(S)$.*

## 4.6 Powers

The integral powers of a real number $x \neq 0$ are defined by induction as follows:

**Definition 25** *Let $x$ be a non-zero real number. If $n \in \boldsymbol{N}$ then*

$$x^1 = x \quad \text{and} \quad x^{n+1} = x \cdot x^n.$$

*Then we set $x^0 = 1$ and $x^{-n} = 1/x^n$.*

Our objective here is to extend the definition to rational powers of a **positive** real number, $x$. To do this we first need to prove:

**Theorem 5** *Let $\boldsymbol{R}_+ = \{x \in \boldsymbol{R} |\ x > 0\}$ be the set of positive real numbers. Then for any natural number $n \in \boldsymbol{N}$:*

1. *For $x, y \in \boldsymbol{R}_+$,*
$$x < y \Leftrightarrow x^n < y^n.$$

2. *The map*
$$\boldsymbol{R}_+ \to \boldsymbol{R}_+, \quad x \mapsto x^n$$
*is a 1-1 correspondence.*

**Proof:** Suppose $x, y \in \mathbf{R}_+$,. The Difference Theorem gives

$$y^n - x^n = (y - x)\Sigma_{i=0}^{n-1} y^i x^{n-1-i}.$$

Since each $y^i x^{n-1-i} > 0$ it follows that $x < y \Leftrightarrow x^n < y^n$. In particular this shows that the map $\mathbf{R}_+ \to \mathbf{R}_+$ given by $x \mapsto x^n$ is 1-1. To prove it is onto we need two Lemmas. We state and prove them next and then use them to complete the proof of the Theorem.

**Lemma 11** *If $x, y \in \boldsymbol{R}_+$ and $y < x^n$, then there is a positive real number $u$ such that*

$$u < x \quad and \quad y < u^n.$$

**Proof:** By hypothesis,

$$0 < \frac{x^n - y}{nx^{n-1}}.$$

Choose $u$ so that $u > 0$ and

$$x - \frac{x^n - y}{nx^{n-1}} < u < x.$$

Then

$$x - u < \frac{x^n - y}{nx^{n-1}}.$$

By the Difference Theorem,

$$x^n - u^n = (x - u)\Sigma_{i=0}^{n-1} u^i x^{n-i-1} < (x - u)nx^{n-1} < x^n - y.$$

Therefore $y < u^n$.
q.e.d.

**Lemma 12** *If $x, y \in \boldsymbol{R}_+$ and $x^n < y$, then there is a positive real number $v$ such that*

$$x < v \quad and \quad v^n < y.$$

**Proof:** Choose $v$ so that $x < v < 2x$ and also

$$v < x + \frac{y - x^n}{n(2x)^{n-1}}$$

Then

$$v - x < \frac{y - x^n}{n(2x)^{n-1}}$$

Since $x < v < 2x$, it follows from the Difference Theorem that

$$v^n - x^n = (v - x)\Sigma_{i=0}^{n-1} x^i v^{n-i-1} < (v - x)nv^{n-1} < (v - x)n(2x)^{n-1} < y - x^n.$$

Therefore $v^n < y$.
q.e.d.

Now we return to the proof of the Theorem, where we have to show that the map

$$\mathbf{R}_+ \to \mathbf{R}_+, \quad x \mapsto x^n$$

is onto. We need to show that if $y$ is a positive real number and $n \in \mathbf{N}$ then there is a positive real number $x$ such that $x^n = y$.

**Fix some $y > 0$ and some $n \in \mathbf{N}$.** We first construct a real number $x$ and second show that $x^n = y$.

To construct $x$ we define a set $T$ by

$$T = \{w \in \mathbf{R}_+ \mid w^n > y\}.$$

Observe that $T \neq \phi$. In fact, since $y > 0$, it follows that $y + 1 > 1$ and so $(y + 1)^{n-1} \geq 1$ and

$$(y + 1)^n = (y + 1)^{n-1}(y + 1) \geq y + 1 > y.$$

Thus $y + 1 \in T$.

On the other hand, $0 < T$ and so $T$ is bounded below. **Therefore $T$ has a greatest lower bound,** $glb(T)$.

Set $glb(T) = x$.

Now we have to show that

$$x^n = y.$$

By Property 3, exactly one of the following three possibilities must hold:

$$x^n > y, \; x^n < y, \quad \text{or} \quad x^n = y.$$

To prove our assertion we show by contradiction that the two inequalities above are false, and so it must be true that $x^n = y$.

But if $x^n > y$, then by Lemma 11 some $u < x$ satisfies $u^n > y$. Thus $u \in T$ and $x$ would not be a lower bound for $T$, contradicting the fact that $x = glb(T)$.

But But if $x^n < y$, then by Lemma 12 some $v > x$ satisfies $v^n < y$. Thus if $w \in T$ then

$$v^n < y \leq w^n,$$

and so $v < w$. Thus $v$ would be a lower bound for $T$, and so $x$ would not be the greatest lower bound, again contradicting our assumption that $x = glb(T)$.

It follows from these two contradictions that $x^n = y$, and this completes the proof of the Theorem.

q.e.d.

**Definition 26** *If $y$ is a positive real number and $n$ is a natural number then the unique positive number $x$ such that $x^n = y$ is called the* ***nth root of*** *$y$ and is denoted by $x = y^{1/n}$.*

This Theorem allows us to construct our first real number that is not rational. It shows that there is a unique positive real number $x$ such that $x^2 = 2$. Thus now we know that $\sqrt{2}$ exists. On the other hand, according to Exercise 3, $\sqrt{2}$ is not a rational number.

We can now define any rational power of a positive real number. For this we need the following:

**Lemma 13** *Suppose $n, q \in \boldsymbol{N}$ and $m, p \in \boldsymbol{Z}$. If $x$ is a positive real number and $m/n = p/q$ then*

$$(x^{1/n})^m = (x^{1/q})^p.$$

**Proof:**: First note that

$$((x^{1/n})^m)^{nq} = (x^{1/n})^{mnq} = ((x^{1/n})^n)^{mq} = x^{mq}.$$

In the same way,

$$((x^{1/q})^p)^{nq} = (x^{1/q})^{qpn} = ((x^{1/q})^q)^{pn} = x^{pn}.$$

Since $m/n = p/q$ it follows that $mq = pn$. Therefore

$$((x^{1/n})^m)^{nq} = ((x^{1/q})^p)^{nq}.$$

By the Theorem above, raising to the $(nq)^{th}$ power is a 1-1 map. Therefore

$$(x^{1/n})^m = (x^{1/q})^p,$$

and the Lemma is proved.
q.e.d.

Now let $a$ be any rational number. We can express $a$ in the form $a = m/n$ with $m \in \mathbf{Z}$ and $n \in \mathbf{N}$. Lemma 13 shows that if $x$ is a positive real number then $(x^{1/n})^m)$ does not depend on the choice of $m$ and $n$. Thus we may make the

**Definition 27** *Let $x$ be any positive real number and let $a$ be any rational number. Then*

$$x^a = (x^{1/n})^m,$$

*where $a = m/n$ and $m \in \boldsymbol{Z}$ and $n \in \boldsymbol{N}$.*

**Exercise 18** ***Powers***

1. *Show that if* $a, b$ *are rational numbers and* $x$ *is a positive real number then* $x^a x^b = x^{a+b}$.
2. *Show that if* $a, b$ *are rational numbers and* $x$ *is a positive real number then* $(x^a)^b = x^{ab}$.
3. *Show that if* $x \in \boldsymbol{R}$ *satisfies* $x > 1$ *and if* $c \in \boldsymbol{Q}$ *is positive, then* $x^c > 1$.
4. *Show that if* $x > 1$ *is a real number and if* $a < b$ *are positive rational numbers then* $0 < x^a < x^b$.
5. *Show that if* $a \in \boldsymbol{Q}$ *is positive and if* $0 < x < y$ *then* $x^a < y^a$.
6. *If* $k \in \boldsymbol{N}$ *and* $x > 0$ *is a real number, show by induction on* $k$ *that*

$$x^k \leq x \quad \textit{if} \quad x < 1 \quad \textit{and} \quad x^k \geq x \quad \textit{if} \quad x > 1.$$

7. *If* $x, \varepsilon \in \boldsymbol{R}$ *are positive and if* $x < 1$ *show that for some* $N \in \boldsymbol{N}$,

$$x^a < \varepsilon \quad \textit{if} \quad a \in \boldsymbol{Q} \quad \textit{and} \quad a \geq N.$$

8. *Let* $p < q \in \boldsymbol{N}$ *and* $x > 0$ *be a real number. For which* $x$ *is* $x^{p/q} > x$ *and for which* $x$ *is* $x^{p/q} < x$*? Prove your answer is correct.*
9. *Let* $x \in \boldsymbol{R}$ *and let* $S$ *be the set of those rationals of the form* $u = p/10^k$ *satisfying the three conditions: (i)* $p \in \boldsymbol{Z}$*, (ii)* $k \in \boldsymbol{N}$*, and (iii)* $u < x$*. Show that* $x = lub(S)$*. Hint: Follow the following steps:*
   - (a) *Let* $\varepsilon > 0$ *be an arbitrary positive number. Use Proposition 6 to conclude that there are rational numbers* $a, b$ *such that* $x - \varepsilon < a < b < x$.
   - (b) *Show that for some* $k \in \boldsymbol{N}, 10^k b - 10^k a > 2$*. Explain why this implies that some integer* $p$ *satisfies* $10^k a < p < 10^k b$.
   - (c) *Conclude that* $p/10^k \in S$ *and that* $x - \varepsilon < p/10^k < x$.

## 4.7 Constructing the Real Numbers

Call a set $S$ of rationals that is bounded above ***full*** if

$$y \in S \text{ and } z < y \text{ implies that } z \in S.$$

Then simply **define** the real numbers to be the subsets of the rationals that are bounded above and full, and identify each rational $p/q$ with the set of all rationals $\leq p/q$. Intuitively we have just "filled in the holes" among the rationals.

If $S$ and $T$ are sets of rationals that are bounded above and full, then we think of these sets as real numbers $x$ and $y$ and define $x < y$ if $S \subsetneq T$, and $x > y$ if $S \supsetneq T$. Then set $x + y$ to be the set

$$S + T = \{p/q + m/n \mid p/q \in S \quad \text{and} \quad m/n \in T\},$$

and define the other algebraic operations in a similar way.

Of course, we then need to prove that all these definitions are well-defined, that all algebraic rules are still true, that the properties above for the order hold, and that the operations and order for the rationals remain as before. And then, of course, we need to prove Properties One through Four. All this is long and boring, so instead we will simply take for granted the existence of the real numbers satisfying those properties.

# Chapter 5

# Infinite Sequences

## 5.1 Convergent sequences

**Definition 28** *An* **infinite sequence** $(x_i)_{i \geq k}$ *of elements in a set* $S$ *is a* ***list*** *of elements* $x_i \in S$*, with* $i \in \boldsymbol{Z}$ *and* $i \geq k$.

*We say the sequence* **begins with** $x_k$*. For any* $p \geq k$*, the infinite sequence* $(x_i)_{i \geq p}$ *is called the pth* **tail** *of the original sequence.*

**Remark:** The set of elements appearing in an infinite sequence **may be finite**, as illustrated by the infinite sequence $1, 0, 1, 0, 1, 0, \ldots$ in which only two integers, 0 and 1 appear.

In the world of applications we never need to know the exact value of a real number - we only need to know it within a given tolerance. Specs for any engineering design always specify heights, lengths, weights etc. up to so many fractions of an inch or so many millimeters, or so many fractions of a gram.

Infinite sequences $(x_i)$ of real numbers are therefore a fundamental tool for applications because they can be used to approximate a real number $x$ to within a given tolerance. Intuitively that means that the error, $|x_i - x|$, gets arbitrarily small as $i$ gets larger. In other words, if we want to approximate $x$ within a given tolerance we may simply use one of the $x_i$ as long as $i$ is sufficiently large.

Infinite sequences are also a fundamental tool in analysis, and for this we need as always to formalize the intuitive idea above:

**Definition 29** *An infinite sequence* $(x_i)_{i \geq k}$ *of real numbers* **converges** *to* $x \in \boldsymbol{R}$ *if* ***for each*** $\varepsilon > 0$ *there is* ***some integer*** $N \geq k$ *(usually depending on* $\varepsilon$*) such that*

$$i \geq N \;\Rightarrow\; |x_i - x| < \varepsilon.$$

**Lemma 14** *If an infinite sequence* $(x_i)$ *of real numbers converges, it converges to a unique real number* $x$.

**Proof:** Suppose the sequence $(x_i)$ converges to both $x$ and $y$.
Then for each $\varepsilon > 0$ there is some $N$ such that for $i \geq N$ **both** $|x_i - x| < \varepsilon/2$ and $|x_i - y| < \varepsilon/2$.

It follows that

$$|x - y| = |x - x_i + x_i - y| \leq |x_i - x| + |x_i - y| < \varepsilon.$$

Since this is true for each $\varepsilon$, $x = y$.
**q.e.d.**

**Definition 30** *If $(x_i)$ is a convergent sequence, the unique $x$ to which it converges is called the* **limit** *of the sequence and is denoted by $lim_i(x_i)$.*

**Example 25** *The infinite sequence $1, 2, 3, \ldots$ of natural numbers does not converge.*

***Proof:*** *This is proved by contradiction. Assume the sequence converges to some $x$. Then for some $N$,*

$$n \geq N \Rightarrow |x_n - x| < 1.$$

*Thus for $n \geq N$,*

$$|x_n| = |x_n - x + x| \leq |x_n - x| + |x| < 1 + |x|.$$

*But by Property Three, for some $k$,*

$$1 + |x| < k < k + N = x_{k+N}.$$

*This is the desired contradiction. q.e.d.*

**Example 26**

1. *The infinite sequence $0, 1, -1, 2, -2, 3, -3, \ldots$ lists all the integers. This sequence does not converge.*

2. *The infinite sequence $(x_n)_{n\geq 1}$ defined by $x_n = 1/n$. This sequence converges to $0$.*

3. *The infinite sequence $(x_n)_{n\geq 1}$ defined inductively by $x_1 = 1$, and*

$$x_n = \sum_{i=1}^{n-1} 2^{x_i}.$$

   *This sequence does not converge.*

4. *The infinite sequence $(x_n)_{n\geq 1}$ defined by: $x_n$ is the nth largest prime number. This sequence does not converge.*

5. *The infinite sequence $(x_n)_{n\geq 1}$ defined by: $x_n = 1+(-1/2)^n$. This sequence converges to 1.*

**Lemma 15** *If $(x_i)$ and $(y_i)$ are infinite sequences of real numbers converging respectively to $x$ and $y$, then $(x_i y_i)$ converges to $xy$.*

**Proof:** If $\varepsilon > 0$, choose $N \in \mathbf{N}$ so that if $n \geq N$ then

$$|x_n - x| < \frac{\varepsilon}{2(|y|+1)}, \quad |y_n - y| < \frac{\varepsilon}{2(|x|+1)},$$

and

$$|y_n - y| < 1.$$

Then for $n \geq N$,

$$|y_n| = |y_n - y + y| \leq |y_n - y| + |y| < 1 + |y|.$$

Therefore for $n \geq N$,

$$|x_n y_n - xy| = |x_n y_n - x y_n + x y_n - xy| \leq |x_n - x||y_n| + |x||y - y_n| < \frac{\varepsilon}{2(|y|+1)}(|y|+1) + |x|\frac{\varepsilon}{2(|x|+1)} \leq \varepsilon.$$

**q.e.d**

**Exercise 19** ***Convergence***
*As always, provide complete proofs for your answers.*

1. *In each of the examples in the second Example above supply proofs for the statements about convergence.*

2. *Suppose $(x_n)_{n\geq 1}$ is an infinite sequence of real numbers converging to $x$. Define the sequence $(y_n)_{n\geq 1}$ by $y_n = x_{2n}$. Is this sequence convergent, and if so, what is its limit? Prove your answer is correct.*

3. *Suppose $(x_n)_{n\geq 1}$ is an infinite sequence of real numbers. Suppose the sequence $y_n = x_{3n}$ converges to $y$ and that the sequence $z_n = x_{3n+2}$ converges to $z$. If $y \neq z$ show that the original sequence $(x_n)$ is not convergent.*

4. *Show that if $x$ is any real number then there is an infinite sequence of rational numbers converging to $x$.*

5. *Suppose $(x_n)_{n\geq 1}$ is an infinite sequence of real numbers converging to $x$. Define a sequence $(y_n)_{n\geq 1}$ by $y_n = x_{n+1} - x_n$. Prove that the sequence $(y_n)_{n\geq 1}$ is convergent, and find its limit. Prove your answer is correct.*

6. *Suppose $(x_i)$ and $(y_i)$ are infinite sequences of real numbers converging respectively to $x$ and $y$.*

   (a) *Show that $(x_i + y_i)$ converges to $x + y$.*

   (b) *If $y \neq 0$ show that for some $n \in \boldsymbol{N}, y_i \neq 0$ for $i \geq n$. In this case prove that the infinite sequence $(1/y_i)_{i\geq n}$ converges to $1/y$.*

7. *If* $(x_i)$ *is a convergent sequence and if each* $x_i \leq b$, *show that* $lim_i(x_i) \leq b$.

8. *If* $(x_i)$ *is a convergent sequence and if each* $x_i \geq a$, *show that* $lim_i(x_i) \geq a$.

9. *If* $(x_i)$ *is an infinite sequence converging to* $x$, *show that the sequence* $(|x_i|)$ *is convergent, and find its limit.*

10. *Suppose* $x > 0$ *and* $(x_i)$ *is an infinite sequence converging to* $x$. *Show that for some* $N \in \boldsymbol{N}$, *that* $x_i > 0,\ i > N$ .

11. *Suppose a sequence* $(x_n)$ *of positive real numbers converges to a positive number. Show that the set* $\{x_n\}$*is bounded below by a positive number.*

12. *For any* $p \in \boldsymbol{N}$ *show that if* $y < x$ *are positive real numbers, then*

$$|y^{1/p} - x^{1/p}| < \frac{|y - x|}{y^{(p-1)/p}}.$$

*(Hint: Use the Difference Theorem)*

13. *If a sequence* $(x_n)$ *of positive real numbers is convergent to* $x$ *and* $x > 0$, *show that for any* $p \in \boldsymbol{N}$*the sequence* $(x_n^{1/p})$ *is convergent to* $x^{1/p}$.

## 5.2 Bounded Sequences

**Definition 31**

1. *An infinite sequence,* $(x_i)_{i \geq k}$, *of real numbers is* **bounded above** *if for some real number* $b$,
$$x_i \leq b \ \text{ for all } \ i \geq k.$$

2. *An infinite sequence,* $(x_i)_{i \geq k}$, *of real numbers is* **bounded below** *if for some real number* $a$,
$$a \leq x_i \ \text{ for all } \ i \geq k.$$

3. *An infinite sequence* $(x_i)_{i \geq k}$ *is* ***bounded*** *if it is both bounded above and bounded below. Thus the sequence is bounded if and only if for some* $a < b$ *and all* $i$,
$$a \leq x_i \leq b.$$

*Equivalently: each* $(x_i$ *is in the closed interval* $[a, b]$.

**Proposition 12** *If* $(x_i)$ *is a convergent sequence in the closed interval* $[a, b]$ *then*

$$lim_i(x_i) \in [a, b].$$

**Proof:** We prove this by contradiction. Let $x = lim_i(x_i)$. If $x > b$ then for some $\delta > 0$ we have

$$x - b > \delta.$$

But since $(x_i)$ cinverges to $x$, for some $i, |x_i - x| < \delta$ and so $x_i - x > -\delta$. Therefore

$$x_i - b = (x_i - x) + (x - b) > -\delta + \delta = 0.$$

This contradicts our hypothesis that $x_i - b \leq 0$.

An identical argument shows that $x \geq a$. q.e.d.

**Definition 32**

1. *An infinite sequence, $(x_i)_{i\geq k}$, of real numbers is* **increasing** *if each $x_i \leq x_{i+1}$.*
2. *An infinite sequence, $(x_i)_{i\geq k}$, of real numbers is* **decreasing** *if each $x_i \geq x_{i+1}$.*

**Lemma 16**

1. *An increasing sequence, $(x_i)_{i\geq k}$ that is bounded above converges, and*

$$lim_i(x_i) = lub\{x_i \mid i \geq k\}.$$

2. *A decreasing sequence, $(y_i)_{i\geq k}$ that is bounded below converges, and*

$$lim_i(y_i) = glb\{y_i \mid i \geq k\}.$$

**Proof:**

1. Set $S = \{x_i \mid i \geq k\}$.
   Then $S$ is bounded above.
   Fix any $\varepsilon > 0$.
   By the definition of $lub(S)$, $lub(S) - \varepsilon$ is **not** an upper bound for $S$.
   Therefore there is some $x_p$ such that

$$x_p > lub(S) - \varepsilon.$$

Choose such a p.

Since the sequence is increasing, for any $i \geq p$,

$$x_i > lub(S) - \varepsilon.$$

On the other hand, since each $x_i \in S$, for $i \geq p$

$$x_i \leq lub(S).$$

Therefore, for $i \geq p$,

$$0 \leq lub(S) - x_i < \varepsilon.$$

In particular,

$$|lub(S) - x_i| < \varepsilon, \quad i \geq p.$$

Since $\varepsilon$ was any positive real number,this is precisely the statement that the sequence converges to $lub(S)$.

2. Set $T = \{y_i \mid i \geq k\}$.
Then $T$ is bounded below.
Set $-T = \{-y_i \mid i \geq k\}$.
Since $T$ is bounded below and multiplication by $-1$ reverses inequalities, $-T$ is bounded above and

$$lub(-T) = -glb(T).$$

Moreover, the sequence $(-y_i)_{i \geq k}$ is increasing.

Thus by the first part of the Lemma, if $\varepsilon > 0$ there is a $p \in \mathbf{N}$ such that

$$|lub(-T) - (-y_i)| < \varepsilon, \quad i \geq p.$$

Since $lub(-T) = -glb(T)$ it follows that $glb(T) = -lub(-T)$ . Therefore, for $i \geq p$,

$$|y_i - glb(T)| < \varepsilon, \quad i \geq p.$$

Thus the sequence $(y_i)_{i \geq k}$ converges to $glb(T)$.

**q.e.d.**

**Exercise 20**

1. *Suppose in a sequence $(x_n)_{n \geq k}$ that for some $N > k$, the sequence $(x_n)_{n \geq N}$ is bounded. Show that the original sequence is bounded.*

2. *Show that convergent sequences are bounded. Hint: Show that if $x_n$ converges to $x$ then for some $N$, $x - 1 < x_n < x + 1$ if $n \geq N$.*

3. *Which of the following infinite sequences are bounded, which are increasing, which are decreasing, and which converge to a limit? If the sequence is convergent, find the limit. As always prove your answers.*

   (a) $1, -1, 1, -1, 1, -1, \ldots$.
   (b) $(x_q)_{q \geq 1} = x^{1/q}$ *where* $0 < x < 1$ *is fixed.*
   (c) $(x_q)_{q \geq 1} = x^{1/q}$ *where* $x = 1$.
   (d) $(x_q)_{q \geq 1} = x^{1/q}$ *where* $x > 1$ *is fixed.*

4. *Suppose $x$ is a real number and that $0 < x < 1$.*

   (a) *Show that the set $S = \{x^n \mid n \in \boldsymbol{N}\}$ is bounded below.*
   (b) *Show that $glb(S) = glb(\{x^n \mid n \geq 2\})$.*
   (c) *Show that $glb(S) = x \cdot glb(S)$, and conclude that $glb(S) = 0$.*
   (d) *Prove that the sequence $(x^n)_{n \geq 1}$ converges and find its limit. Prove that this limit is correct.*

5. *Provide a detailed proof of the assertion $x \geq a$ in Prop 12.*

## 5.3 The Cauchy Criterion for Convergence

**Proposition 13** *A sequence $(x_n)_{n\geq 1}$ of real numbers converges if and only if for each $\varepsilon > 0$ there is some integer $N$ such that $|x_n - x_m| < \varepsilon$ for all $n, m \geq N$.*

**Proof:** *Suppose first that the sequence converges to $x$.*
Then for any $\varepsilon > 0$, there is some $N$, such that

$$|x_n - x| < \varepsilon/2 \ \text{ if } \ i \geq N.$$

Thus if $n, m \geq N$,

$$|x_n - x_m| = |x_n - x + x - x_m| \leq |x_n - x| + |x_m - x| < \varepsilon.$$

Conversely, suppose that for each $\varepsilon > 0$ there is some integer $N$ such that

$$|x_n - x_m| < \varepsilon \ \text{ if } \ n, m \geq N.$$

Define sets $S_n$ by setting

$$S_n = \{x_i \mid i \geq n\}.$$

*We first show that $S_1$ is bounded.* By hypothesis, for some $K$, if $n \geq K$, then

$$|x_n - x_K| < 1.$$

Thus for $n \geq K$,

$$|x_n| \leq |x_n - x_K| + |x_K| < 1 + |x_K|.$$

Since for $n < K$, $|x_n| \leq Max\{|x_1|, \cdots, |x_{K-1}|\}$, it follows that for all $n$,

$$|x_n| < 1 + |x_K| + Max\{|x_1|, \cdots, |x_{K-1}|\}.$$

Thus $S_1$ is bounded.

Now, since each $S_n \subset S_1$, it follows that each $S_n$ is bounded.
Thus we may set

$$a_n = glb(S_n).$$

Since each $S_{n+1} \subset S_n$, it follows that $a_n$ is a lower bound for $S_{n+1}$. But $a_{n+1}$ is the greatest lower bound for $S_{n+1}$. Therefore

$$\cdots \leq a_n \leq a_{n+1} \leq \cdots$$

In other words, $(a_n)$ is an increasing sequence.

Moreover,

$$a_n \leq x_n < 1 + |x_K| + Max\{|x_1|, \cdots, |x_{K-1}|\}.$$

Thus the sequence $(a_n)$ is bounded above. Apply Lemma 12 to conclude that the sequence $(a_n)$ is convergent to $a = lub\{a_n\}$.

*Finally, we show that the sequence* $(x_n)$ *converges to* $a$. Fix any $\varepsilon > 0$ and choose $N$ so that

$$|a_N - a| < \varepsilon/3 \text{ and } \quad |x_n - x_m| < \varepsilon/3 \ \text{ if } \ n, m \geq N.$$

Since $a_N$ is the glb for $S_N$ it follows that for some $m \geq N$ we have

$$|x_m - a_N| < \varepsilon/3$$

Therefore, for $n \geq N$,

$$|x_n - a| \leq |x_n - x_m| + |x_m - a_N| + |a_N - a| < \varepsilon.$$

In other words, the sequence $x_n$ converges to $a$. **q.e.d.**

## 5.4 The Intersection Theorem

We now come to a result, which again uses Property Four about the reals. To state the Theorem we recall the

**Notation:** If $a \leq b$ are real numbers, then $[a, b]$ denotes the finite closed interval

$$[a, b] = \{x \in \mathbf{R} \mid a \leq x \leq b\}.$$

**Theorem 6** *(Intersection Theorem) Let*

$$S_m \supset S_{m+1} \supset S_{m+2} \supset \cdots \supset S_k \supset \cdots$$

*be an infinite sequence of finite closed intervals:*

$$S_k = [a_k, b_k].$$

*Then*

1. $(a_k)$ *is an increasing sequence bounded above, and convergent to* $a = lub\{a_k\}$*;*
2. $(b_k)$ *is a decreasing sequence bounded below, and convergent to* $b = glb\{b_k\}$*;*
3. $b - a \geq 0$.
4. $(b_k - a_k)$ *is a decreasing sequence of non-negative numbers convergent to* $b - a$*;*
5. $$\bigcap_k [a_k, b_k] = [a, b].$$

**Proof:** By definition, $a_k \leq b_k$. Since $[a_k, b_k] \supset [a_{k+1}, b_{k+1}]$ it follows that $a_k$ is a lower bound for $S_{k+1}$ and $b_k$ is an upper bound for $S_{k+1}$ But by definition, $a_{k+1}$ is the greatest lower bound for $S_{k+1}$ and $b_{k+1}$is the least upper bound for $S_{k+1}$. Therefore

$$a_k \leq a_{k+1} \leq b_{k+1} \leq b_k.$$

Thus $(a_k)$ is an increasing sequence and $(b_k)$ is a decreasing sequence.

Moreover, since

$$a_k \leq b_k \leq b_1,$$

the sequence $(a_k)$ is both increasing and bounded above. Therefore this sequence converges to $a = lub\{a_k\}$.

Similarly,

$$a_1 \leq a_k \leq b_k.$$

Thus the sequence $(b_k)$ is decreasing and bounded below. It follows that the sequence $(b_k)$ converges to $b = glb\{b_k\}$.

In particular the sequence $(b_k - a_k)$ converges to $b - a$, and since each $b_k - a_k \geq 0$, it follows that $b - a \geq 0$.

It remains to show that

$$\bigcap_k [a_k, b_k] = [a, b].$$

But if $x \in [a, b]$ then for all $k$,

$$a_k \leq a \leq x \leq b \leq b_k,$$

and so $x \in [a_k, b_k]$. Thus $x \in \bigcap_k [a_k, b_k]$; i.e.,

$$\bigcap_k [a_k, b_k] \supset [a, b].$$

On the other hand, suppose $x \in \bigcap_k [a_k, b_k]$. Then for all $k$

$$a_k \leq x \leq b_k.$$

Thus $x$ is an upper bound for the set $\{a_k\}$ and a lower bound for the set $\{b_k\}$. It follows that

$$x \geq a \quad and \quad x \leq b.$$

Thus $x \in [a, b]$, and

$$\bigcap_k [a_k, b_k] \subset [a, b].$$

This completes the proof of the Theorem.
**q.e.d.**

**Corollary** In the Theorem, suppose $lim_k(b_k - a_k) = 0$. Then

$$\bigcap_k [a_k, b_k]$$

is a single point $c$. If $(x_k)$ is an infinite sequence with $x_k \in [a_k, b_k]$ then the sequence $(x_k)$ converges to $c$.

**Proof:** It follows from the Theorem that $b - a = 0$; $i.e., b = a$.
Denote this point by $c$.
Then $[a, b] = \{c\}$, and so

$$\bigcap_k [a_k, b_k] = \{c\}.$$

Now, let $\varepsilon > 0$ be any positive number.
Then choose $N$ so that $b_k - a_k < \varepsilon$ for $k \geq N$.
Since $c \in [a_k, b_k]$ and $x_k \in [a_k, b_k]$ it follows that

$$k \geq N \;\Rightarrow\; |x_k - c| < \varepsilon.$$

Thus the sequence $(x_k)$ converges to $c$.
**q.e.d.**

**Exercise 21** *In this exercise, $n \in \boldsymbol{N}$.*

1. *Construct a sequence of open intervals $I(n) = (a_n, b_n)$ such that the sequence $(b_n - a_n)$ converges to zero, and such that each $I(n+1) \subset I(n)$, but such that*
$$\bigcap_n I(n) = \phi.$$

2. *Construct a sequence of closed intervals $I(n) = [a_n, b_n]$ such that the sequence $(b_n - a_n)$ converges to zero, and such that each $I(n+1) \bigcap I(n)) \neq \phi$, but such that*
$$\bigcap_n I(n) = \phi.$$

3. *Let $S_n$ be the set of rational numbers $a$ satisfying $\sqrt{2} - 1/n < a < \sqrt{2} + 1/n$. Find $\cap_n S_n$.*

4. *Let $S_n$ be the sets in the previous problem. If $x_n \in S_n$, does the sequence $(x_n)$ converge? If so, find its limit.*

5. *Let $S_n$ be the set of non-zero numbers in $[-1/n, 1/n]$. What is the intersection of these sets?*

6. *Let $S_n$ be the open interval $(0, 1 + 1/n)$. What is the intersection of these sets?*

## 5.5 Subsequences

**Definition 33** *A* **subsequence** *of a sequence* $(x_n)_{n\geq k}$ *is a sequence* $(y_i)_{i\geq r}$*, defined by*

$$y_i = x_{n_i}$$

*in which* $n_r < n_{r+1} < n_{r+2} \dots$ *is an infinite increasing sequence of integers.*

**Exercise 22** ***Subsequences***

1. *Construct a sequence which has two convergent subsequences converging to different limits.*

2. *Construct a sequence which for each* $n \in \boldsymbol{N}$ *has a subsequence converging to* $n$*.*

3. *Show that any subsequence of a convergent sequence converges to the same limit.*

4. *Show that a sequence which is not bounded above has an increasing subsequence which is not bounded above.*

5. *Let* $(x_n)$ *be a sequence such that the set* $S = \{x_n\}$ *is finite. Show that there is a subsequence* $(x_{n_i})$ *such that for all* $i, x_{n_i} = x_{n_1}$*.*

**Theorem 7** *Every bounded sequence contains a convergent subsequence.*

**Proof:** Let $(x_n)_{n\geq 1}$ be a bounded sequence, and let $S = \{x_n \mid n \geq 1\}$.
Since the sequence is bounded there are numbers $a_1 < b_1$ such that

$$S \subset [a_1, b_1].$$

We first construct by induction on $n$ a decreasing sequence of intervals

$$[a_1, b_1] \supset [a_2, b_2] \supset \cdots \supset [a_n, b_n] \cdots$$

with the following two properties:

1. Each interval $[a_n, b_n]$ contains $x_k$ for infinitely many $k$, and

2. $$b_n - a_n = \frac{b_1 - a_1}{2^{n-1}}.$$

By hypothesis, $x_k \in [a_1, b_1]$ for all $k$.

Now suppose by induction that the intervals $[a_i, b_i]$ are constructed for $i \leq n$. By construction $[a_n, b_n]$ contains $x_k$ for infinitely many $k$. Then there are two mutually exclusive possibilities: either

$$[a_n, \frac{a_n + b_n}{2}] \text{ contains } x_k \text{ for infinitely many } k,$$

or

$$[a_n, \frac{a_n + b_n}{2}] \text{ contains } x_k \text{ for only finitely many } k.$$

In the second case,

$$[\frac{a_n + b_n}{2}, b_k]$$

must contain $x_k$ for infinitely many $k$.

In the first case set

$$a_{n+1} = a_n \quad \text{and} \quad b_{n+1} = \frac{a_n + b_n}{2}.$$

In the second case set

$$a_{n+1} = \frac{a_n + b_n}{2} \quad \text{and} \quad b_{n+1} = b_n.$$

By our induction hypothesis $b_n - a_n = (b_1 - a_1)/2^{n-1}$, and therefore

$$b_{n+1} - a_{n+1} = \frac{b_n - a_n}{2} = \frac{b_1 - a_1}{2^n}.$$

This completes the inductive construction.

Our second step is to construct by induction a subsequence $(x_{n_i})$ so that for all $i$:

$$x_{n_i} \in [a_i, b_i].$$

First, set $x_{n_1} = x_1$. Then

$$x_{n_1} = x_1 \in S \subset [a_1, b_1]$$

Then suppose by induction that the $x_{n_i}$ are constructed for $i \leq r$.
Since $[a_{r+1}, b_{r+1}]$ contains $x_k$ for infinitely many $k$ there is a least integer $p > n_r$ such that $x_p \in [a_{r+1}, b_{r+1}]$.
Set $x_{n_{r+1}} = x_p$. This completes the inductive construction of the subsequence.

It remains to show that this subsequence is convergent.
Since

$$b_r - a_r = \frac{b_1 - a_1}{2^{r-1}},$$

it follows that the sequence $b_r - a_r$ converges to zero. Thus by the Corollary to the Theorem, since $x_{n_r} \in [a_r, b_r]$, the sequence $(x_{n_r})$ converges. **q.e.d.**

# Chapter 6

# Continuous Functions of a Real Variable

## 6.1 Real-valued Functions of a Real Variable

Functions from the reals to the reals are a central tool in almost every discipline that uses mathematics. Among the many examples are:

- Position and speed of a particle as a function of time.

- Crop yield as a function of total precipitation.

- Grade on an exam as a function of time spent studying.

- Cost to the national health system as a function of the amount of pollution in the air.

- Fraction of the population that is illiterate as a function of the average time per pupil spent in class on reading and writing.

- Wave length of light reaching us from a star as a function of its distance away.

As you may easily imagine, functions of a real variable are also a core part of mathematics itself. In fact this field is a prime example of how mathematics interacts with other disciplines: many of the problems and theorems in mathematics in this area are inspired by questions and phenomena from outside, while the results and techniques that mathematicians discover frequently get applied elsewhere.

This chapter focuses on two key concepts in the analysis of functions: limits and continuity. But first we establish some basic definitions.

**Definition 34** ***Intervals***

1. *An* **interval** *is a subset of* $\boldsymbol{R}$ *of one of the following forms (where* $a$ *and* $b$ *are any real numbers):*
   (a) $\boldsymbol{R}$
   (b) $[a,b],\ (a,b],\ [a,b)$ *or* $(a,b)$
   (c) $[a,\infty),\ (a,\infty),\ (\infty,b]$ *or* $(\infty,b)$
2. *The intervals in (b) are called* **finite***; the first is* **closed***, the next two are* **half closed***, and the fourth is* **open***.*
3. *The numbers* $a$ *and* $b$ *are called the* **end points** *of their intervals.*
4. *The* **closure** *of an interval* $D$ *is the union of* $D$ *together with any end points. It is denoted by* $\overline{D}$*.*

**Remark: In general, mathematicians study maps $D \to S$ from a general domain $D \subset \mathbf{R}$ to a general target $S \subset \mathbf{R}$. However, in this introduction to real-valued functions we shall limit ourselves to the case that $D$ is an interval.**

**Definition 35** *In this class, by a* **real-valued function** *we shall mean a set map*

$$f : D \to S$$

*from an interval* $D$ *to a subset* $S \subset \boldsymbol{R}$*.* $D$ *is called the* ***domain*** *of* $f$ *and* $S$ *is its* ***target****.*

**Definition 36** *If* $f : D \to S$ *is a real-valued function and if* $E$ *is a second interval contained in* $D$ *then the* **restriction** *of* $f$ *to* $E$ *is the real function* $g : E \to S$ *defined by* $g(x) = f(x),\ x \in E$*.*

**Example 27** ***Real-valued functions***

1. *If* $f : D \to S$ *and* $g : D \to S$ *are real-valued functions then* $f + g : D \to \boldsymbol{R}$ *and* $fg : D \to \boldsymbol{R}$ *are the functions defined by*

$$(f+g)(x) = f(x) + g(x) \quad \textit{and} \quad (fg)(x) = f(x)g(x).$$

2. *If* $f : D \to S$ *is a real-valued function and if* $f(x) \neq 0$ *for all* $x \in D$ *then* $1/f : D \to \boldsymbol{R}$ *is the function defined by*

$$(1/f)(x) = 1/f(x).$$

3. $f : \boldsymbol{R} \to \boldsymbol{R}$ *defined by*

$$f(x) = \begin{cases} 0, & x \in \boldsymbol{Q}, \\ 1, & x \notin \boldsymbol{Q}. \end{cases}$$

4. $f : (0, \infty) \to \boldsymbol{R}$ *defined by:* $f(x) = 1/x$.

5. $f : (0, 1) \to \boldsymbol{R}$ *defined by:* $f(x)$ *is the number in the* $25th$ *decimal place of* $x$.

6. 
$$f(x) = \begin{cases} 0, & x < 0, \\ 1, & x \geq 0. \end{cases}$$

7. $f(x) = \sum_{i=0}^{n} \lambda_k x^k$, *where the* $\lambda_k$ *are real numbers and* $\lambda_n \neq 0$. *These functions are the* ***real polynomials of degree*** $n$.

## 6.2 Limits

Suppose $f : D \to S$ is a real-valued function. An important question which arises is:

**Can we use the values of** $f(x)$ **near a point** $c \in \overline{D}$ **to determine a "limiting value"**?

There is a good practical reason for this question. In practice one can never make an exact measurement at a specific point, $c$. The best one can do is make measurements nearby and hope they give a good approximation to a measurement at $c$ itself.

This concept has **two fundamental ingredients**:

1. We consider only values of the function **near** $c$ and **not** the value at $c$.
2. The values of the function at points near $c$ must **bunch ever more closely together** as the points get closer to $c$.

This idea is formalized in the following way:

**Definition 37** *Suppose* $f : D \to S$ *is a real-valued function defined in an interval* $D$, *and that* $c \in \overline{D}$. *Then*

$$\mathbf{f(x) \to u} \textit{ as } \mathbf{x \to c}$$

*if for* ***any*** $\varepsilon > 0$ *there is* ***some*** $\delta > 0$ *such that*

$$x \in D \quad \textit{and} \quad 0 < |x - c| < \delta \Rightarrow |f(x) - u| < \varepsilon.$$

**Note:** If you are asked to prove that $\mathbf{f(x) \to u}$ as $\mathbf{x \to c}$ then:

1. you are **given** some arbitrary positive $\varepsilon$ about which **all you know** is that it is positive,

2. you have to **show** there is some positive $\delta$ such that this condition holds.

Usually $\delta$ will **depend** on $\varepsilon$.

**Lemma 17** *Suppose $f : D \to S$ is a real-valued function defined in an interval $D$. If for some $c \in \overline{D}$, as $x \to c$*

$$f(x) \to u_1 \quad \textit{and} \quad f(x) \to u_2,$$

*then $u_1 = u_2$.*

**Proof:** Fix $\varepsilon > 0$. Choose $\delta_1 > 0$ and $\delta_2 > 0$ so that for $x \in D$ ,

$$|f(x) - u_1| < \varepsilon/2 \text{ if } 0 < |x - c| < \delta_1, \text{ and}$$

$$|f(x) - u_2| < \varepsilon/2 \text{ if } 0 < |x - c| < \delta_2.$$

Set $\delta$ to be the lesser of $\delta_1$ and $\delta_2$. Then for $x \in D$ and $0 < |x - c| < \delta$ we have

$$|u_2 - u_1| = |u_2 - f(x) + f(x) - u_1| \leq |u_2 - f(x)| + |f(x) - u_1| < \varepsilon.$$

Thus $|u_2 - u_1| < \varepsilon$ for all $\varepsilon > 0$ and so $u_2 = u_1$. **q.e.d.**

**Definition 38** *If $f(x) \to u$ as $x \to c$, then $u$ is the **limit** of $f(x)$ as $x$ approaches $c$ and we write*

$$\boldsymbol{lim_{x \to c} f(x) = u}.$$

**Important Remarks:**

1. The definition of limit specifies the condition

$$\mathbf{0 < |x - c| < \delta}.$$

   Thus the limit depends on the values of $f(x)$ when $x$ is close to but **different from** $c$. Even when $c$ is in the interval $D$ the value of $f(c)$ is **irrelevant** to the definition of $lim_{x \to c} f(x)$.

2. If $f : D \to S$ is a real-valued function and $c \in \overline{D}$, then there are **two** possibilities, and **either could be correct**:

   (a) The limit of $f(x)$ as $x \to c$ **exists**, or

   (b) This limit does **not** exist.

3. If $f : D \to S$ is a real-valued function and $c \in D$ then there are **three** possibilities:

   (a) The limit of $f(x)$ as $x \to c$ exists and $lim_{x\to c}f(x) = f(c)$, or

   (b) The limit of $f(x)$ as $x \to c$ exists and $lim_{x\to c}f(x) \neq f(c)$, or

   (c) The limit of $f(x)$ as $x \to c$ does not exist.

**Example 28** *Limits*

1. *$D = \boldsymbol{R}, S = \boldsymbol{R}$ and $f(x) = 5x + 1$. Then*

$$lim_{x\to 1}f(x) = 6.$$

   *In fact given $\varepsilon > 0$ set $\delta = \varepsilon/5$. If $0 < |x - 1| < \delta$, then*

$$|f(x) - 6| = 5|x - 1| < 5(\varepsilon/5) = \varepsilon.$$

2. *$D = S = \boldsymbol{R}$ and*

$$f(x) = \begin{cases} 5x + 1, & x \neq 1, \\ 17, & x = 1. \end{cases}$$

   *Then, as above, $lim_{x\to 1}f(x) = 6$, but $f(1) = 17$.*

3. *$D = (-\infty, 1), S = \boldsymbol{R}$ and $f(x) = 5x + 1$. Then*

$$lim_{x\to 1}f(x) = 6.$$

   *(Same proof as above.)*

## 6.3 Limits and Negations

Recall that the **negation** of the statement "A is true" is the statement "A is false". Thus one way of showing that a statement is true is to show that its negation is false! And often we do this by showing that the negation leads to a contradiction. **For this it is often helpful to restate the negation in a more useful form.**

**Example 29** *The negation of Statement A: $lim_{x\to c}f(x) = u$.*

*So what does it mean for Statement A to be false?*

*If Statement A is true then for each $\varepsilon > 0$ there is a $\delta > 0$ satisfying*

$$x \in D \quad and \quad 0 < |x - c| < \delta \Rightarrow |f(x) - u| < \varepsilon.$$

*Therefore to say Statement A is false is precisely to say there must be **some** $\varepsilon > 0$ so that **no** $\delta > 0$ will work for that particular $\varepsilon$.*

*Now what does it mean for $\delta$ to work for that $\varepsilon$?*

*It means precisely that for **every** $x \in D$ such that $0 < |x - c| < \delta$ it is true that $|f(x) - u| < \varepsilon$.*

*Thus to say that **no** $\delta > 0$ will work for that $\varepsilon$ means that for each $\delta > 0$ there is some $x \in D$ such that*

$$0 < |x - c| < \delta \quad \textit{and} \quad |f(x) - u| \geq \varepsilon.$$

*Altogether then we have:*

***Statement A** is false if and only if Statement B below is true:*

***Statement B:** for some $\varepsilon > 0$ and every $\delta > 0$ there is some $x \in D$ such that $0 < |x - c| < \delta$ and $|f(x) - u| \geq \varepsilon$.*

*Statement B is the **negation** of Statement A.*

**Example 30** $D = S = \boldsymbol{R}$ *and*

$$f(x) = \begin{cases} 1/|x|, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

*In this example $lim_{x \to 0} f(x)$ does **not** exist.*

We have to prove that **no** $u \in \mathbf{R}$ is the limit of $1/|x|$ as $x \to 0$.

Fix any $u \in \mathbf{R}$. We have to show that there is **some** $\varepsilon > 0$ such that:

for **every** $\delta > 0$ there is some $x \in \mathbf{R}$ such that

$$0 < |x| < \delta \quad \text{and} \quad |1/|x| - u| > \varepsilon.$$

We can restate this as follows: there is some *varepsilon* $> 0$ and some $x$ in $\mathbf{R}$ such that: for any $\delta > 0$,

$$0 < |x| < \delta \quad \text{and} \quad |1/|x| - u| > \varepsilon.$$

First, choose $\varepsilon = 1$. Then choose $x$ so that

$$|x| < \delta \quad \text{and} \quad 0 < |x| < \frac{1}{|u| + 1}.$$

Then $1/|x| > |u| + 1$. Therefore

$$|1/|x| - u| \geq 1/|x| - |u| > 1 + |u| - |u| \geq 1 = \varepsilon.$$

Therefore $f(x)$ does not have a limit as $x \to 0$.

## 6.4 Limits of Sequences and Limits of Functions

In the previous chapter we introduced the limit of an infinite sequence, and now we have another kind of limit. It is natural to wonder if these two kinds of limit are connected, and indeed they are. The next Proposition shows how.

**Proposition 14** *Let $f : D \to S$ be a real-valued function defined in an interval $D$. Suppose that*

$$c \in \overline{D}.$$

*Then*

$$lim_{x \to c} f(x) = u \Leftrightarrow \; lim_n f(x_n) = u$$

*for each sequence $x_n \in D - c$ converging to $c$.*

**Proof:** We have two prove two things:

1. **If** $lim_{x \to c} f(x) = u$ **then** $lim_n f(x_n) = u$ whenever $(x_n)$ is a sequence of points in $D$ different from $c$ but convergent to $c$.

2. **If** $lim_n f(x_n) = u$ whenever $(x_n)$ is a sequence of points in $D$ different from $c$ but convergent to $c$, **then** $lim_{x \to c} f(x) = u$.

For the first statement our hypothesis is $lim_{x \to c} f(x) = u$, and we have to prove that if $(x_n)$ is any sequence of points in $D - c$ converging to $c$, then

$$lim_n f(x_n) = u.$$

In other words, given $\varepsilon > 0$ we need to show that for some $N$,

$$|f(x_n) - u| < \varepsilon \quad \text{if} \quad n \geq N.$$

Since by hypothesis $lim_{x \to c} f(x) = u$, it follows that for some $\delta > 0$,

$$|f(x) - u| < \varepsilon \text{ if } 0 < |x - c| < \delta.$$

Since the sequence $(x_n)$ converges to $c$ and no $x_n = c$ there is some $N \in \mathbf{N}$ for which

$$0 < |x_n - c| < \delta \;\text{ if }\; n \geq N.$$

Therefore $|f(x_n) - u| < \varepsilon$ if $n \geq N$. This proves that the sequence $f(x_n)$ converges to $u$.

For the second statement our **hypothesis** is that whenever $(x_n)$ is a sequence of points in $D$ different from but converging to $c$ then

$$lim_n f(x_n) = u,$$

and **we need to prove that**

$$lim_{x \to c} f(x) = u.$$

**We prove this by showing that the negation of this statement is false**, and so the statement must be true.

As described in the previous section, the negation is Statement B:

For **some** $\varepsilon > 0$ and **every** $\delta > 0$ there is some $x \in D$ such that:

$$0 < |x - c| < \delta \ \text{ and } \ |f(x) - u| \geq \varepsilon.$$

To prove Statement B is false we assume it is true and deduce a contradiction. Now if Statement B is true, then for each $n \in \mathbf{N}$ there is some point $x_n \in D$ such that

$$0 < |x_n - c| < \frac{1}{n} \ \text{ and } \ |f(x_n) - u| \geq \varepsilon.$$

This defines a sequence $(x_n)$ of points in $D$ different from $c$.

First we show that this sequence converges to $c$. Indeed,for any real number $\sigma > 0$ there is some $N \in \mathbf{N}$ such that $1/N < \sigma$ (Property Three for the real numbers). Therefore, for $n \geq N$

$$|x_n - c| < 1/n \leq 1/N < \sigma.$$

Thus the sequence $(x_n)$ converges to $c$.

Now our hypothesis states that since $(x_n)$ converges to $c$, it follows that $f(x_n)$ converges to $u$. But we constructed the sequence so that

$$|f(x_n) - u| \geq \varepsilon$$

for all $n$, which contradicts this hypothesis.

It follows that Statement B cannot be correct and therefore that $lim_{x\to c} f(x) = u$.

**q.e.d.**

**Exercise 23** ***Limits***

1. *If $c < d$ are points in an interval $D$, show that $[c, d] \subset D$.*
2. *Suppose $f : (a, b) \to \boldsymbol{R}$ is a real-valued function and that for some $u \in \boldsymbol{R}$, $f(x) < u$ for all $x \in (a, b)$. If $lim_{x\to b} f(x)$ exists, show that*

$$lim_{x\to b} f(x) \leq u.$$

3. *For each of the following choices of $D$, $f$, and $c$, and with $S = \boldsymbol{R}$ decide if $lim_{x\to c} f(x)$ exists and when it does find the limit.*
   (a) $D = (-1, 0)$, $c = 0$, $f(x) = 1, x \in D$.

(b) $D = [0, 1),\ c = 0,$ *and* $f(x) = 0, x \in D$.

(c) $D = (-1, 1),\ c = 0,$ *and*

$$f(x) = \begin{cases} 1, & x \in (-1, 0], \\ 0, & x \in (0, 1). \end{cases}$$

4. *Suppose* $f : D \to \boldsymbol{R}$ *and* $g : D \to \boldsymbol{R}$ *are real-valued functions in an interval* $D$. *If* $c \in \overline{D}$ *and if* $lim_{x\to c}f(x)$ *and* $lim_{x\to c}g(x)$ *exist, show that*

$$lim_{x\to c}(f(x) + g(x)) = lim_{x\to c}f(x) + lim_{x\to c}g(x), \quad \text{and}$$

$$lim_{x\to c}f(x)g(x) = (lim_{x\to c}f(x))(lim_{x\to c}g(x)).$$

*In particular conclude that these limits exist.*

5. *Suppose* $f : D \to S$ *is a real-valued function in an interval* $D$. *Suppose further that for some* $c \in D$ *and some* $\alpha > 0$,

$$f(x) \neq 0 \quad \text{if} \quad 0 \leq |x - c| < \alpha.$$

*Show that if* $lim_{x\to c}f(x)$ *exists and* $lim_{x\to c}f(x) \neq 0$, *then*

$$lim_{x\to c}\frac{1}{f(x)}$$

*exists.*

6. *Suppose* $f : \boldsymbol{R} \to \boldsymbol{R}$ *is a polynomial. Show that for all* $y \in \boldsymbol{R}$,

$$lim_{x\to y}f(x) = f(y).$$

7. *Suppose* $f : D \to \boldsymbol{R}$ *is a real-valued function in an interval,* $c \in \overline{D}$, *and* $z \in \boldsymbol{R}$,. *Assume that for each* $0 < \varepsilon < 1$ *there is a* $\delta > 0$ *such that if* $0 < |x - c| < \delta$ *then* $|f(x) - z| < \varepsilon$. *Show that* $lim_{x\to c}f(x) = z$.

8. *Suppose* $x, y$ *are positive real numbers and* $n \in \boldsymbol{N}$.

   (a) *Show that if* $y \in \boldsymbol{R}$ *and* $y > 0$ *then* $lim_{x\to y}x^{1/n} = y^{1/n}$. *(Hint: See Exercise 18.12.)*

9. *Do the following limits exist?*

$$lim_{x\to 0}sin(\frac{1}{x}) \quad \text{and} \quad lim_{x\to 0}xsin(\frac{1}{x}).$$

*You may use the following facts:*

(a) $-1 \leq sin(y) \leq 1$ *for all* $y \in \boldsymbol{R}$.

*(b)*

$$sin(\frac{\pi k}{2}) = \begin{cases} 0, & k = 2n, \\ 1, & k = 4n+1, \\ -1 & k = 4n-1. \end{cases}$$

10. *Define a real-valued function $f : \boldsymbol{R} \to \boldsymbol{R}$ in the interval by*

$$f(x) = \begin{cases} 0, & x < 0, \\ 1, & x \geq 0. \end{cases}$$

*For which $c \in \boldsymbol{R}$ does $lim_{x \to c} f(x)$ exist?*

11. *Let $n \in \boldsymbol{N}$ and define a function $f : (-\infty, a) \to \boldsymbol{R}$ by*

$$f(x) = \frac{x^n - a^n}{x - a}.$$

*Show that $lim_{x \to a} f(x)$ exists, find the limit and prove your answer is correct.*

## 6.5 Continuous Functions

Let $f : D \to S$ be a real-valued function defined in an interval, $D$. If $c \in D$ and if $lim_{x \to c} f(x)$ exists then we have two numbers:

$$lim_{x \to c} f(x) \quad \text{and} \quad f(c).$$

As we saw in the previous exercises, these two numbers may be different!

However, **if** these numbers are the same **then** we can approximate the **value at** $c$ of $f$ by the **values nearby** $c$. Functions for which these two numbers are the same have many good properties, and play a crucial role in mathematics. In fact they have a name:

**Definition 39** *A real-valued function $f : D \to S$ in an interval $D$ is* **continuous** *if for each $c \in D$ the limit $lim_{x \to c} f(x)$ exists and*

$$lim_{x \to c} f(x) = f(c).$$

**Lemma 18** *Suppose $f : D \to S$ is a continuous function in an interval $D$. Then for each $\varepsilon > 0$ and $c \in D$ there is a $\delta > 0$ such that*

$$x \in D \quad and \quad |x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon.$$

**Remark:** The $\delta$ in the Lemma **will usually depend on both** $\varepsilon$ and $a$.

**Proof:** Since $f$ is continuous, $lim_{x \to c} f(x) = f(c)$. In particular, the limit exists.

Fix any $\varepsilon > 0$. By the definition of limit, there is some $\delta > 0$ such that if $x \in D$ and $0 < |x - c| < \delta$, then

$$|f(x) - f(c)| < \varepsilon.$$

But this inequality is trivially true if $x = c$.
**q.e.d.**

**Proposition 15** *A function $f : D \to S$ in an interval $D$ is continuous if and only if whenever a sequence $(x_n)$ of points in $D$ converges to $c \in D$ then the sequence $(f(x_n))$ converges to $f(c)$.*

**Remark:**This Proposition states that a function is continuous if and only if it preserves convergent sequences.

**Proof:** First, suppose $f$ is continuous. Choose $\delta > 0$ so that

$$|f(x) - f(c)| < \varepsilon \quad \text{if} \quad x \in D \quad \text{and} \quad |x - c| < \delta.$$

Since the sequence $(x_n)$ in our hypothesis converges to $c \in D$, it follows that for some $N$,

$$|x_n - c| < \delta \quad \text{if} \quad n \geq N.$$

Thus for $n \geq N$,

$$|f(x_n) - f(c)| < \varepsilon.$$

Therefore $(f(x_n))$ converges to $f(x)$.

On the other hand suppose $f$ preserves convergent sequences. If $c \in D$ then for any sequence $(x_n)$ of points in $D$ converging to $c$, by hypothesis

$$lim_n f(x_n) = f(c).$$

It follows from Proposition 14 in the previous section that $lim_{x \to c} f(x) = f(c)$.
**q.e.d.**

**Exercise 24** ***Continuous functions***

1. *Show that if $\lambda \in \boldsymbol{R}$ the constant function $\boldsymbol{R} \to \lambda$ is continuous.*
2. *If $f : D \to S$ and $g : D \to S$ are continuous functions in an interval $D$, show that the functions $f + g$ and $fg$ are continuous.*
3. *If $f : D \to S$ is a continuous function in an interval $D$, and if $f(x) \neq 0$ for all $x \in D$, show that $1/f$ is continuous.*
4. *Show that polynomials are continuous functions in $\boldsymbol{R}$.*
5. *If $n \in \boldsymbol{N}$ show that $x^{1/n}$ is a continuous function in $(0, \infty)$.*
6. *Show that sin $x$ and cos $x$ are continuous functions. (You may use the trigonometric formulae for $sin(x + y)$ and $cos(x + y)$.)*

7. *If $f : D \to \boldsymbol{R}$ is a continuous function in an interval $D$ and if $g : \boldsymbol{R} \to T$ is a continuous function in $\boldsymbol{R}$, show that the composite $g \circ f$ is continuous.*

8. *If $f : D \to S$ is a continuous function defined in an interval $D$, show that the restriction of $f$ to a second interval contained in $D$ is continuous.*

9. *If $f : D \to S$ is a continuous function defined in $D = [a, b)$ and if $lim_{x \to b} f(x) = u$, show that a continuous function $g : [a, b] \to \boldsymbol{R}$ is defined by*

$$g(x) = \begin{cases} f(x), & x \in [a, b), \\ u, & x = b. \end{cases}$$

10. *Suppose $f : D \to S$ is a continuous function in an interval $D$.*

    (a) *Show that the function $|f| : D \to \boldsymbol{R}$ defined by $|f|(x) = |f(x)|$ is continuous.*

    (b) *Show that the functions $f_+ : D \to \boldsymbol{R}$ and $f_- : D \to \boldsymbol{R}$ defined by*

    $$f_+(x) = \begin{cases} f(x), & f(x) \geq 0, \\ 0, & f(x) < 0. \end{cases} \quad \textit{and}$$

    $$f_-(x) = \begin{cases} -f(x), & f(x) \leq 0, \\ 0, & f(x) > 0. \end{cases}$$

    *are continuous. (Hint: Consider $1/2(f + |f|)$.)*

    (c) *Show that $f = f_+ - f_-$.*

Recall that a continuous function is a real-valued function whose value at a given point is the limit of the values at points approaching the given point. Continuous functions in a closed interval automatically satisfy a stronger condition:

**Proposition 16** *Suppose $f : D \to S$ is a continuous function defined in a closed interval $D = [a, b]$. Then for all $\varepsilon > 0$ there is some $\delta > 0$ such that*

$$|f(y) - f(x)| < \varepsilon \quad \textit{if} \quad x, y \in [a, b] \quad \textit{and} \quad |x - y| < \delta.$$

**Remark:** Note that the $\delta$ in this Proposition **does not depend** on $x$.

**Proof of Proposition 16**: We assume the Proposition is false and derive a contradiction.

If the Proposition is false there is some $\varepsilon > 0$ such that for each $\delta > 0$ there is a correspnding pair of points $x, y \in [a, b]$ satisfying

$$|f(x) - f(y)| \geq \varepsilon \quad \text{and} \quad |x - y| < \delta.$$

In particular, we may find sequences $(x_n)$ and $(y_n)$ of points in $[a, b]$ such that

$$|f(x_n) - f(y_n)| \geq \varepsilon \quad \text{and} \quad |x_n - y_n| < \frac{1}{n}.$$

Since the sequences $(x_n)$ and $(y_n)$ are bounded the sequence $(x_n)$ contains a convergent subsequence $(u_i) = (x_{n_i})$. In the same way the sequence $(v_i) = (y_{n_i})$ contains a convergent subsequence $(v_{i_k})$. As a subsequence of a convergent sequence, the subsequence $(u_{i_k})$ is also convergent.
Set

$$lim_k(u_{i_k}) = x \;\; \text{and} \;\; lim_k(v_{i_k}) = y.$$

Since the $x_n$ and the $y_n$ are in $[a, b]$, it follows that $x \in [a, b]$ and $y \in [a, b]$.

Since $|x_n - y_n| < 1/n$ it follows that

$$lim_k|u_{i_k} - v_{i_k}| = 0,$$

and so $x = y$.

Since $f$ is continuous it preserves convergent sequences and so

$$f(x) = lim_k(f(u_{i_k}) = lim_k(f(v_{i_k}) = f(y).$$

But this is impossible because

$$|f(y_n) - f(x_n)| \geq \varepsilon$$

for all $n$.

This is the desired contradiction.
**q.e.d.**

## 6.6 Continuous Functions Preserve Intervals

In this section we establish an important and fundamental property of continuous functions.

**Theorem 8** *Suppose $f : D \to S$ is a continuous function defined in a closed interval $D = [a, b]$. Then the image of $f$ is a closed interval:*

$$f([a, b]) = [c, d]$$

*for some real numbers $c \leq d$.*

**Consequences of Theorem 8.**

1. **Maximum and minimum values.** Since $[c, d]$ is the image of $f$ there must be points $u, v \in [a, b]$ such that
$$f(u) = c \text{ and } f(v) = d.$$
Thus
$$f(u) \leq f(x) \leq f(v)$$
for all $x \in [a, b]$.

2. **Intermediate values.** Since $[c, d]$ is the image of $f$ it follows that for every $y \in [c, d]$ there is some $x \in [a, b]$ such that
$$f(x) = y.$$

**Important Remarks:** Let $f : D \to S$ be a continuous function defined in the interval $D = [a, b]$.

1. Then the Theorem states that
   (a) The set of values of $f$ is bounded.
   (b) $f$ has a minimum and maximum value.
   (c) Every number between the minimum and maximum value is a value of $f$.

2. However, the points $u, v$ where $f$ attains its minimum and maximum values may be anywhere inside the interval $[a, b]$, and it may well happen that $v < u$!

**Example 31**

*1.* $[a, b] = [-1, 1]$ *and*
$$f(x) = \begin{cases} x, & x \leq 0, \\ -2x, & x \geq 0. \end{cases}$$
*Here the image of $f$ is the interval $[-2, 0]$ and $f$ attains its minimum at 1 and its maximum at 0.*

*2. $f$ is defined in $(0, 1)$ by $f(x) = 1/x$. Here the values of $f$ are not bounded above but are bounded below by 1; however, $f$ does not attain a minimum value since*
$$glb\{f(x)\} = 1$$
*and 1 is not a value of $f$.*

While the properties above are consequences of the Theorem we actually have to prove them first and then use them to prove the Theorem! We do this now.

**Proposition 17** *Suppose $f : D \to S$ is a continuous function defined in a closed interval $D = [a, b]$. Then for some $u, v \in [a, b]$,*

$$f(u) \leq f(x) \leq f(v)$$

*for all $x \in [a, b]$.*

**Proof:** Set $S = \{f(x) \mid x \in [a, b]\}$. We show first by contradiction that $S$ is bounded above.

In fact, if $S$ is not bounded above then for each $n \in \mathbf{N}$ there is an $x_n \in [a, b]$ such that $f(x_n) > n$. Since $x_n$ is a bounded sequence it contains a convergent subsequence $(x_{n_i})$. By Proposition 12 the limit $z$ of this subsequence is in $[a, b]$. Since continuous functions preserve convergent sequences the sequence $(f(x_{n_i}))$ is convergent. But

$$f(x_{n_i}) > n_i$$

and so this sequence is not bounded above. Therefore it cannot be convergent. This is a contradiction, and so $S$ must be bounded above.

Since $S$ is bounded above, for each $n$ there is some $y_n \in S$ such that

$$0 \leq lub(S) - y_n < \frac{1}{n}.$$

In particular, $lim_n(y_n) = lub(S)$.

On the other hand, because $y_n \in S$ we may write $y_n = f(x_n)$ for some $x_n \in [a, b]$. Then $(x_n)$ is a bounded sequence. Therefore there is a subsequence $x_{n_k}$ converging to some point $v \in [a, b]$. Because continuous functions preserve convergent sequences the sequence $f(x_{n_k})$ converges to $f(v)$.

But $f(x_{n_k})$ is a subsequence of the convergent sequence $f(x_n)$. Therefore this subsequence has the same limit. Hence

$$f(v) = lim_k f(x_{n_k}) = lim_n f(x_n) = lim_n y_n = lub(S).$$

In other words, for all $x \in [a, b]$,

$$f(x) \leq f(v).$$

The proof of the existence of $u$ is identical.
**q.e.d.**

**Definition 40** *The points $u, v$ in Proposition 15 are called respectively an* **absolute minimum point** *and an* **absolute maximum point** *for $f$.*

**Proposition 18** *If $f : D \to S$ is a continuous function defined in an interval $D$ and if $f(x) < f(y)$ for some $x, y \in D$, then for any $w$ with $f(x) < w < f(y)$ there is some $z \in D$ with $f(z) = w$.*

**Proof:** We argue by contradiction, and so suppose that there is no $z \in D$ with $f(z) = w$.

Let $d = |y - x|$. We construct by induction a sequence of intervals $I_n \subset D$, $n \geq 0$ with endpoints $x_n$ and $y_n$ such that

$$|y_n - x_n| = \frac{d}{2^n}, \quad and$$

$$f(x_n) < w < f(y_n).$$

(Note: it may happen that $x_n < y_n$ or that $y_n < x_n$!!)

First set $x_0 = x$ and $y_0 = y$.

Now suppose $x_n$ and $y_n$ are constructed. Since $w$ is not a value, either

$$f(\frac{x_n + y_n}{2}) < w \quad or \quad f(\frac{x_n + y_n}{2}) > w.$$

In the first case, set

$$x_{n+1} = \frac{x_n + y_n}{2} \quad and \quad y_{n+1} = y_n.$$

In the second case, set

$$x_{n+1} = x_n \quad and \quad y_{n+1} = \frac{x_n + y_n}{2}.$$

In either case

$$|y_{n+1} - x_{n+1}| = \frac{d}{2^{n+1}}, \quad and$$

$$f(x_{n+1}) < w < f(y_{n+1}).$$

Now by the Intersection Theorem, $\bigcap I_n$ is a single point $c$. Moreover, the Corollary to that Theorem states that

$$lim_{n\to\infty} x_n = c = lim_{n\to\infty} y_n.$$

Since $f$ is continuous, it follows that

$$lim_{n\to\infty} f(x_n) = f(c) = lim_{n\to\infty} f(y_n).$$

But $f(x_n) < w$ for all $n$, and thus $f(c) \leq w$. Also $f(y_n) > w$ for all $n$, and thus $f(c) \geq w$.

It follows that $f(c) = w$, contradicting our hypothesis that $w$ was not a value of $f$. **q.e.d.**

**Proof of Theorem 8**: By Proposition 17, $f$ has an absolute minimum at some $u \in [a, b]$ and an absolute maximum at some $v \in [a, b]$:

$$f(u) \leq f(x) \leq f(v)$$

for all $x \in [a, b]$. If $w \in [f(u), f(v)]$, then either $w = f(u)$ or $w = f(v)$ or $f(u) < w < f(v)$. But in this third case it follows from Proposition 18 that $w = f(x)$ for some $x \in [a, b]$.

**Exercise 25**

*In these exercises you may assume the standard properties of sin x and cos x.*

1. *Let $f : D \to S$ be any real-valued function defined in an interval $D$.*
   (a) *Show that if $f$ attains a maximum then that maximum is $lub(S)$ where $S = \{f(x) \mid x \in D\}$.*
   (b) *If $f$ attains a minimum value, what is it?*