Some Precursor Inequalities

Kolmogorov Inequality (Loève, 1955, Sec. 16.2.A) If the independent random variables ξ_k for $k \geq 1$ are integrable, then

$$P\Big(\max_{k \leq n} \Big| \sum_{j=1}^{k} (\xi_j - E(\xi_j))\Big| \geq t\Big) \leq \frac{1}{t^2} \sum_{k=1}^{n} \mathrm{Var}(\xi_k)$$

used in 3-series Theorem. Next: restriction used with truncation

Kolmogorov (upper) Exponential Bounds (Loève Sec. 18.1.A): if $\xi_{k,n}$ are indep with all $|\xi_{k,n}| \leq c$:

$$P\Big(\sum_{k=1}^{n} \xi_{k,n} \Big/ \Big[\sum_{k=1}^{n} \mathrm{Var}(\xi_{k,n})\Big]^{1/2} > \epsilon\Big) \leq \begin{cases} \exp(-\frac{\epsilon^2}{2}(1 - \epsilon c/2)) & \text{if } \epsilon c \leq 1 \\ \exp(-\epsilon/(4c)) & \text{if } \epsilon c \geq 1 \end{cases}$$

Used with lower bounds in proving Law of Iterated Logarithm.

Martingales and Sub-Martingales

A few main sources for martingales and submartingales:

(1) Martingale: $X_t = \sum_{s=1}^{t} \xi_s$, with $E(\xi_t \,|\, X_1, \ldots, X_{t-1}) = 0$ (or ≥ 0, for submartingale)

(2) Martingale: $X_t = E(Y \,|\, Z_1, \ldots, Z_t)$, eg, $Y = dQ/dP$

(3) Submartingale: $X_t = A_t + \epsilon_t$, with $A_t \nearrow$ meas. with respect to $X_1, \ldots, X_{t-1}$, and $E(\epsilon_t \,|\, \mathcal{F}_{t-1}^{X}) = 0$ Doob Decomp.

(4) Submartingale: $X_t = g(Y_t)$, Y_t a martingale, g convex.

Conditional Jensen Ineq.: $E\{g(Y_t) \,|\, \mathcal{F}_{t-1}^{Y}\} \geq g\Big(E\{Y_t \,|\, \mathcal{F}_{t-1}^{Y}\}\Big)$

Max-Inequality Generalization to (Sub-)Martingales

Doob 1953 Inequality: For submartingale X_t with $E|X_t| < \infty$, then $P(\sup_{0 \le t \le T} X_t \ge 1) \le E(X_T^+) \le E|X_T|$

General Reference: A. DasGupta (2008) **Asymptotic Theory of Statistics and Probability** (Springer, free pdf from library)

includes Burkholder L_p norm $\|X_t\|_p$ (upper and lower) inequalities in terms of $\|\{\sum_{s=1}^t (X_s - X_{s-1})^2\}^{1/2}\|_p$

Other maximal inequalities available (Chow 1960; Binbaum-Marshall 1961; Lenglart 1977; Slud 1986) in discrete and continuous time to bound $X_t \cdot h_{t+}$ above, for X_t submartingale and $h_t \in \mathcal{F}_{t-}$ non-negative nonincreasing.

Submartingale Inequality & Chernoff Method

So far, we did not restrict distributions or mgf's of increments.

if X_n is a martingale with sub-exponential (ν_s, α_s) increments $\xi_s = X_s - X_{s-1}$ given $\mathcal{F}^X_{s-1}$, then inductively for $|\lambda| < 1/\alpha_s$ and $|\mu| < 1/\alpha_* = 1/\max_{s \le n} \alpha_s$, by repeated conditioning

$$E(e^{\lambda \xi_s} \,|\, \mathcal{F}_{s-1}) \le e^{\lambda^2 \sigma_s^2/2} \;\Rightarrow\; E\Big(\prod_{s=1}^n e^{\mu \xi_s}\Big) \le e^{\mu^2 \sum_{s=1}^n \sigma_s^2/2}$$

implies via the Doob maximal inequality:

$$P(\max_{0 \le s \le n} (X_s - X_0) \ge t) \;\le\; \exp\Big[\inf_{\mu < 1/\alpha_*} (\frac{\mu^2}{2} \sum_{s=1}^n \sigma_s^2 \;-\; \mu t)\Big]$$

Thus gives maximal-inequality version of Azuma-Hoeffding, with $\alpha_s = 0$, $\sigma_s = c$, for $|\xi_s| \le c$.

Wainwright Cor. 2.21 on 'Bounded Differences'

Say $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, and x' is a k-*neighbor* if it differs only in the k'th coordinate. Say $f : \mathbb{R}^n \to \mathbb{R}$ has Bounded Differences (BDP) if $|f(x) - f(x')| \le L_k$, for all such x, x', k.

Corollary 1 *If f has BDP and $X = (X_1, \ldots, X_n)$ has indep. components, then*

$$P(|\, f(X) - Ef(X)\,| \ \ge\ t) \ \le\ \exp\Big(-2t^2 / \sum_{k=1}^{n} L_k \Big)$$

Idea: consider martingale differences $\xi_k = E(f(X)\,|\,X_1, \ldots, X_k) - E(f(X)\,|\,X_1, \ldots, X_{k-1})$, bounded between $-L_k$, L_k.

BDP Corollary, continued

Main step:

$$\inf_x \; E(f(X) \,|\, X_1, \ldots, X_k)\Big|_{X_k = x} - E(f(X) \,|\, X_1, \ldots, X_{k-1}) \;\geq\; -L_k$$

by BDP, similarly with $\sup_x \leq L_k$.

Note: 'maximal' version would bound the tail probability for

$\max_k E(f(X) \,|\, X_1, \ldots, X_k) - E(f(X))$