Stat 410 Sample Problems for Test 2

Instructions. The test could consist of 4 or 5 problems of the level of difficulty and rough extent of coverage of these. The answers (e.g., for (4)) need not be given as decimal numbers but must be written out to the point where only simple arithmetic would be needed to get from your expression to a numerical answer. **The test will be closed-book.**

(1). *(Binomial approximation to normal)* A coin is flipped independently in 400 batches of 5 tosses each. Assume that the probability of heads on each toss is $2/5$. Find approximately the probability that the total number of heads resulting in Heads falls between 770 and 820 inclusive.

(2). *(Probability calculation & independence, joint continuous)* Suppose that the joint density of continuous random variables X, Y is given for a constant parameter $\lambda > 0$, by

$$f_{X,Y}(x, y) \;=\; \lambda\, y^{-1}\, e^{-\lambda y}\, I_{[0<x<y]}$$

(a) Find the marginal density of Y, and the conditional density of X given Y, and show that these random variables are not independent.

(b) Find $P(X < Y/3)$.

(3). *(Combinatorial joint probabilities, discrete r.v.'s)* Suppose that a fair die is tossed repeatedly and independently. Define X as the number of the toss on which a 4 first appears, and Y as the number of the toss on which a 3 first appears. Are X and Y independent ? Find the joint probability mass function of X and Y, and calculate $P(X \geq 2Y)$.

(4). *(Univariate probabilities and moments)* Let

$$f(x) \;=\; c\, x^{\gamma-1}\, e^{-(x/\beta)^{\gamma}}\, I_{[x>0]}$$

where $\beta,\ \gamma > 0$ are positive parameters.

(a) Find the value of c which makes f a density.

(b) Compute a formula in terms of $k,\ \beta,\ \gamma$ for $E(X^k),\ k > 0$.

(c) What is $Var(X)$ when $\beta = 1,\ \gamma = 2$?

(5). *(Univariate Transformation, & Properties of Expectation)* Let the continuous random variable V have density function $f_V(v) = 4\,v^3\,I_{[0<t<1]}$.

(a) Find the variance of $V + V^2$.

(b) Find the density function of $(1 + V)^{-3}$.

Note that you should **not** *calculate the density of the varianble in (a) in order to find its first and second moments.*

(6). *(Relations between types of variables)* What kind of random variable is Y in each of the following, i.e., what is its named type of density ? Specify the parameter values in each case.

(a) Y = sum of independent variables X_1, X_2, X_3, where X_i are independent Gamma(2,2) r.v.'s.

(b) $Y = X_1 + X_2$, where X_i are independent normally distributed random variables with $E(X_i) = i$ and $E(X_i^2) = 6$ for $i = 1, 2$.

(c) $Y = 3 + 2 \cdot X$, where $X \sim \mathcal{N}(1, 4)$.

(d) $Y = 3T^2$, where $T \sim$ Weibull$(2, 3)$ has density $f_T(t) = 6t\,e^{-3t^2}\,I_{[t>0]}$. *Hint: look at the distribution function of* Y.

(7). *(Convolution)* Find the density of $Y = V + W$, where $V \sim$ Expon(1) and $W \sim$ Expon(2) are independent.

(8). *(Covariance and Correlation)* Suppose that (X, Y) is a pair of random coordinates jointly distributed uniformly in the triangle $\{(x, y) : \; 0 \le x \le 1, \; 0 \le y \le x\}$. Find the mean of X, the mean of Y, and the covariance of X, Y.

(9). *(Multivariate Change of Variable)* Suppose that X, Y are independent random variables jointly uniformly distributed on the unit disk $\{(x, y) : x^2 + y^2 \le 1\}$. Find the joint density of the polar coordinates (R, Θ) of these two variables defined by the relations

$$X \;=\; R\,\cos\Theta \quad , \qquad Y \;=\; R\,\sin\Theta$$

Are R and Θ independent ?