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Sample Problems for Stat 650 In-Class Test

The following 8 problems cover topics that are central to our Stat 650 course material so far. (Lecture on Monday will give detail relevant to the ones on Exponential waiting-time probabilities and ODE's in continuous-time chains.) All of these problems have fairly short answers, and you should make sure you understand them well. The test will consist of three (or maybe a choice of 3 out of 4) such problems. On the test, as here, you should expect primary emphasis and basic Markov chain definitions, concepts amd theoretical results, and at least one question requiring you to find a numerical probability or expectation using first-step analysis.

(1). Suppose that a discrete-time HMC X_n with finite or countable state-space S has an invariant probability distribution $\pi = (\pi_j,\ j \in S)$, with $\pi^{tr} P = \pi^{tr}$. Assume also that the chain is irreducible.

(a). Show from first principles that $\pi_j > 0$ for all $j\ inS$.

(b). Is it true that $P^{(n)}_{jk}$ tends to a limit for all j, k as $n \to \infty$? If so, what is that limit ?

(2). In the homogeneous-transition discrete-time Markov chain X_n with states $S = \{0, 1, 2\}$ and

$$\mathbf{P} \;=\; \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

(a). Find $E(T_0 \,|\, X_0 = 0)$, where $T_0 = \inf\{n \geq 1 :\ X_0 = 0\}$.

(b). By symmetry between states 1 and 2, necessarily $E(T_0 \,|\, X_0 = 1) = E(T_0 \,|\, X_0 = 2)$. Find this expectation.

(3). Let $\mathbf{P}$ be the transition matrix for a countable-state HMC. Let $\mathbf{P}^*$ be defined from $\mathbf{P}$ by

$$P^*_{jk} \;=\; \frac{1}{2}\, P_{jk} \,+\, \frac{1}{2}\, I_{[k=0]} \qquad \text{for all} \quad j, k \in S$$

Show that if a chain X^*_n with this transition matrix $\mathbf{P}^*$ is irreducible, then it is positive-recurrent.

(4). Consider a continuous-time HMC with states $\{0, 1, 2\}$ and embedded transition matrix given by $P_{1,2} = P_{1,0} = 1/2, \quad P_{0,1} = P_{2,1} = 1$, with other transition probabilities 0, and with all transition-rates $\lambda_j = 1$ for transition away from states j. Give an ordinary differential equation satisfied by $P_{11}(t)$, and solve it.

(5). Identify the closed classes (equivalence classes with respect to the relation of mutual-communication) for a HMC X_n with states $\{1, 2, \ldots, 6\}$ and thetransition matrix P given below. Also tell for which k the limits of $P_{1k}^{(n)}$ are positive when $n \to \infty$.

$$\mathbf{P} \;=\; \begin{pmatrix} 1/4 & 0 & 1/2 & 1/4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/3 & 1/3 & 0 & 1/3 \\ 1/4 & 0 & 1/4 & 1/2 & 0 & 0 \\ 0 & 2/3 & 0 & 0 & 1/3 & 0 \\ 0 & 1/3 & 0 & 0 & 1/3 & 1/3 \end{pmatrix}$$

(6). Find $P(X_1 > \min(X_2, 2X_4) + X_3)$, where the random variables are independent and distributed as $X_i \sim \text{Expon}(\lambda_i)$ with density $\lambda_i \exp(-\lambda_i x)$, $x > 0$, for $i = 1, 2, 3, 4$.

(7). Find $x = P_1(\text{hit } \{3, 4\} \text{ before } \{5, 6\})$ in the discrete-time HMC with states $\{1, 2, \ldots, 6\}$ and transitions $p_{j,k} = 1/4$ for $k - j \bmod 6 \in \{0, 1, 2, 3\}$ and $= 0$ for other j, k.

(8). Suppose that the transition probabilities for a HMC on $S = \{0, 1, 2, \ldots\}$ are given by $p_{0,1} = 1, \quad p_{j,j+1} = 1 - p_{j,0} = j/(j+1)$ for $j \geq 1$. We showed in an earlier exercise (and do not show it now!) that if the chain starts at $X_0 = 0$ with probability 1, then the return-time random variable T_0 takes the value $j + 1$ with probability $1/(j(j+1)) = 1/j - 1/(j+1)$.

(a). Show using this fact that the chain is **null-recurrent**.

(b). Explain briefly why (a) implies for fixed k , as $n \to \infty$, $P_{0k}^{(n)} \to 0$.

(c). Explain why (b) implies that $E_0(X_n) \to \infty$ as $n \to \infty$.