STAT 750 Handout on Spherical Symmetry

We saw in class that a joint density on $\mathbb{R}^p$ is spherically symmetric (equivalently, that a random vector with this density is *rotationally invariant* if and only if for every $p \times p$ orthogonal matrix U, the identity $f(Ux) = f(x)$ holds for x in a set of probability 1. Equivalently, this holds if and only if $f(x)$ has the form $h((x_1^2 + \cdots + x_p^2)^{1/2})$.

We made a claim in class and sketched a proof idea that for X a rotationally invariant random vector, $R = (X_1^2 + \cdots + X_p^2)^{1/2} = \|X\|_2$ and $Z = X/R$ are independent, with R a scalar-valued random variable with density proportional to $h(r)$ on $(0, \infty)$ and X/R uniformly distributed on the surface of the unit sphere in $\mathbb{R}^p$. We provide two proofs of this.

First, let's define the concept of a uniformly distributed random unit-length vector Z in $\mathbb{R}^p$. Intuitively, that should mean for each open subset B of the surface S of the unit sphere in $\mathbb{R}^p$,

$$P(Z \in B) = \text{vol}_{p-1}(B)/\text{vol}_{p-1}(S)$$

where $\text{vol}_{p-1}(\cdot)$ is the calculation volume within the $p-1$-dimensional surface of the unit sphere (the 'surface area' when $p = 3$ and the circumference length-measure when $p = 2$). With respect to this $p-1$-dimensional surface volume measure, the random unit vector Z is called **uniform** if it has constant density, and that constant must be $1/\text{vol}_{p-1}(S)$. We evaluate that constant in the next paragraphs of this handout.

Because each small open set $B \subset S$ centered around a single unit-vector z can be regarded as a volume-element that can be carried into a volume element centered around Uz by an orthogonal matrix U, it follows immediately from the rotational invariance of X that for any orthogonal U,

$$P(R \in (r, r+\epsilon),\ Z \in B) \ = \ P(R \in (r, r+\epsilon),\ Z \in UB)$$

The collection of sets UB, by choice of small B, covers the whole spherical surface S and can be used to define or approximate Riemann-type integrals over S. By covering S with tiny sets UB with very small overlap, we conclude also that $P(Z \in B \,|\, R \in (r \in (r, r+\epsilon))$ depends on B only through its volume and therefore that R and Z are independent. That is a sketch-proof of the independence and the uniform distribution of Z. The p-dimensional set $\{x :\ \|x\|_2 \in (r, r+\epsilon)\}$ is easily seen to be the relative complement $\{x : \|x\|_2 \le r+\epsilon\} \setminus \{x :\ \|x\|_2 \le r\}$, and its volume is equal to the volume of the whole unit sphere $\{x :\ \|x\|_2 \le 1\}$ times $(r+\epsilon)^p - r^p$. Assume for convenience that $f(x)$ and $h(r)$ are continuous functions, and let $C_p = \text{vol}_p(\{x \in \mathbb{R}^p : \ \|x\|_2 \leq 1\}$. Then our argument shows that for small ϵ,

$$P(R \in (r, r+\epsilon)) \ = \ \int_{\{x:\ \|x\|_2 \in (r, r+\epsilon)\}} h(\|x\|_2)\, dx \ = \ C_p\, h(r)\, p\, r^{p-1}\, \epsilon \ + \ o(\epsilon)$$

and it follows that the density of $R = \|X\|_2$ for $r \in (0, \infty)$ is $p\, C_p\, h(r)$. On the other hand, the set $\{x : \ \|x\|_2 \in (r, r+\epsilon)\}$ evidently also has volume $\epsilon\, r^{p-1} \cdot \text{vol}_{p-1}(S) + o(\epsilon)$, so our argument proves also

$$\text{vol}_{p-1}(S) \ = \ p\, \text{vol}_p(\{x : \ \|x\|_2 \leq 1\}) \ = \ p \, \cdot \, C_p$$

The constants in the last paragraph do not depend on the form of the function $h(r)$ and can therefore be evaluated in the special case where X has *iid* $\mathcal{N}(0,1)$ components. Then

$$h(r) \ = \ (2\pi)^{-p/2}\, e^{-r^2/2} \quad \Longrightarrow \quad 1 \ = \ \int_{\mathbb{R}^p} h(\|x\|_2) dx \ = \ \int_0^\infty p\, C_p\, r^{p-1}\, (2\pi)^{-p/2}\, e^{-r^2/2}\, dr$$

It follows by the change-of-variables $w = r^2/2$ that

$$1 \ = \ p\, C_p\, (2\pi)^{-p/2} \int_0^\infty (2w)^{p/2-1}\, e^{-w}\, dw \ = \ p\, C_p\, (2\pi)^{-p/2}\, 2^{p/2-1}\, \Gamma(p/2)$$

and therefore $C_p = \pi^{p/2}/\Gamma((p+2)/2)$. These formulas agree with the common formulas for area $C_2 = \pi$ and circumference $2C_2 = 2\pi$ when $p = 2$, and for volume $C_3 = 4\pi/3$ and surface-area $\text{vol}_2(S) = 3 \cdot C_3 = 4\pi$ when $p = 3$

Finally, we provide a more formal proof of the independence of R, Z using a slight extension of the Jacobian change-of-variable formula. Suppose that $q : \mathbb{R}^p \to \mathbb{R}^p$ is a differentiable k-to-1 map such that there are K disjoint subsets A_k partitioning $\mathbb{R}^p$ (which means $\cup_{k=1}^K A_k = \mathbb{R}^p$) and differentiable inverse maps $\psi_k : \mathbb{R} \to A_k$ such that $\psi_k \circ q : A_k \to A_k$ is the identity-map. Then if $X \sim f(x)$ is a random p-vector, the random vector $V = q(X) \in \mathbb{R}^p$ has probability density

$$f_V(v) \ = \ \sum_{k=1}^{K} f \circ \psi_k(v)\, |det(J_{\psi_k}(v))| \tag{1}$$

where as usual $J_{\psi_k}(v)$ denotes the Jacobian matrix.

We apply this formula to the mapping $q(x) = (r, z_1, \ldots, z_{p-1})$ where $r = \|x\|_2$ and $(z_1, \ldots, z_p) = x/r$ (which can be defined to be the 0-vector when $x = 0$. Here $K - 2$, and the sets A_k are

$$A_1 = \{x \in \mathbb{R}^p : \;\; x_p \geq 0\} \quad , \qquad A_2 = \{x \in \mathbb{R}^p : \;\; x_p < 0\}$$

and the mappings ψ_k for $k = 1, 2$ are:

$$\psi_k(r, z_1, \ldots, z_{p-1}) \;=\; (rz_1,\, rz_2\, \ldots,\, rz_{p-1},\, (-1)^{k-1}\, r\, \sqrt{1 - z_1^2 - \cdots z_{p-1}^2}\,) \tag{2}$$

Direct calculation of partial derivatives shows that the absolute determinants of the Jacobians J_{ψ_k} are the same and are equal to $r^{p-1}\, \sqrt{1 - z_1^2 - \cdots z_{p-1}^2}$. It follows from (1) that the joint density of $(R, Z_1, \ldots, Z_{p-1})$ is $r^{p-1}\, h(r)\, \sqrt{1 - z_1^2 - \cdots z_{p-1}^2}$. This is not particularly convenient as a way to prove the uniform distribution of $Z = (Z_1, \ldots, Z_p)$ on S, but the factorization of the joint density does prove that R is independent of $(Z_1, \ldots, Z_{p-1})$. By definition, $z_p = (-1)^{k-1}\, \sqrt{1 - z_1^2 - \cdots z_{p-1}^2}$ on A_k for $k = 1, 2$. Since the density of $(R, Z_1, \ldots, Z_{p-1})$ is identical on A_1, A_2, we find that $I_{[Z_p \geq 0]}$ is independent of $(R, Z_1, \ldots, Z_{p-1})$. Thus the three variables R and $(Z_1, \ldots, Z_{p-1})$ and $I_{[Z_p \geq 0]}$ are independent, which implies that R is independent of $(Z_1, \ldots, Z_p)$.

Putting together what we have shown above, we have justified the claim made in class that the most convenient way to simulate a uniform unit-vector Z on the surface of the p-dimensional unit sphere is to simulate the *iid* $\mathcal{N}(0, 1)$ entries of a random multivariate-normal vector X and define $Z = X/\|X\|_2$.