Hints and Comments on HW1

In all these problems, you may find your task easier if you slightly restrict generality, e.g., by assuming that a density is piecewise continuous. That is OK by me.

MKB # 2.7.1. In this problem, write out the two assumptions (the spherical symmetry and the independence of components) as separate equations that the density must satisfy. Then by using both together, try to deduce the most restrictive equation you can that the density must satisfy. This will lead to to (a form of) a famous *functional equation* that will help you complete the problem. If you get that far, I will tell you the name of the functional equation, and from that point Wikipedia or other standard sources will take you the rest of the way if you cannot figure it out from first principles.

Extra Problem (A). In (i), the most important step is to get a formula for (the p-length subvectors of) a typical column of the matrix $V_A \otimes V_B$. This matrix is in the form of a $p \times p$ blockwise array of $p \times p$ matrices. Try to find an expression for the p p-vectors that make up the j'th column of the k'th column of $V_A \otimes V_B$, where j, k are any two integers from $\{1, \ldots, p\}$. Then apply the matrix $A \otimes B$ to that column.

Extra Problem (B). Part (i) is correct, but its proof is a little more subtle than I intended. The *det* function is continuous on the Euclidean space of unrestricted $p \times p$ matrices (which is easy to show from the definition or the formula for expanding determinants by minors and co-factors. But the difficult issue is when a continuous function g applied to a random variable $X \in \mathbb{R}^d$ with a density may have probability atoms. What follow are two paragraphs of remarks and then a strong hint.

Of course 0 is a continuous function, so a continuous function $g(X)$ of a random vector X may always be degenerate, with $Y = g(X)$ assigning all its probability mass to the value 0. More generally, letting X be a real-valued random variable with positive density on $[0, 3]$, the continuous function $Y = g(X) = \max(1, \min(X, 2))$ produces a scalar-valued random variable with probability atoms at $1, 2$, i.e., with positive probability that Y is equal to either 1 or 2. In higher-dimensional examples, the requirement that the random vector X has a density plays an important role. Without that restriction, let $Y = (Y_1, Y_2)$ with Y_i independent $\mathcal{N}(0, 1)$ random variables and L be a Bernoulli(1/2) random variable independent of Y, i.e., $P(L = 0) = P(L = 1) = 1/2$. Then define a continuously distributed random vector

$$X = (X_1, X_2) = I_{[L=1]}\,(Y_1, -Y_1) + I_{L=0]}\,(Y_1, Y_2)$$

For this random vector, the continuous function $g(x_1, x_2) = x_1 + x_2$ takes on the value 0 with probability 1/2.

Could an example like the last one occur if the random variable X were assumed to have a density ? Not if the conditional distribution of X_1 given X_2 has a density. That suggests that we use a conditional density to ensure that when X has a density on $\mathbb{R}^p$, the random variable defined by the continuous function $Y = \det(X)$ does not have a probability atom at 0.

Here is the hint, which has two parts. First, the (signed) eigenvalues $\lambda(A) = \lambda = (\lambda_1, \ldots, \lambda_p)$ of a $p \times p$ matrix A in order of decreasing real part have the property for any real number c that $\lambda(A + cI_{p\times p}) = c + \lambda(A)$, and as argued in class $det(A) = \prod_{j=1}^{p} \lambda_j(A)$. Then second, if the entries of the $p \times p$ random matrix $X = (X_{ij})_{i,j=1}^{p}$ have a joint density $f(x)$, then the conditional distribution of $trace(X) = \sum_{i=1}^{p} X_{ii}$ given $X - (trace(X)/p)I_{p\times p}$ has a density. Put these facts together and you can prove that conditionally given $X - (trace(X)/p)I_{p\times p}$, and therefore also unconditionally, X is nonsingular (equivalently, $det(X) \neq 0$) with probability 1.