STAT750 HW1 Problems.

Read Chapters 1 and 2 of the Mardia, Kent and Bibby (MKB) text. Then do the following 7 problems to be handed in (uploaded) by midnight Monday Feb. 7 on ELMS. (For problems (A) and (B), see also the Matrix Algebra appendix of MKB.

Do MKB problems # 2.5.1, 2.6.4, 2.7.1 and 4 additional problems (A), (B), (C) and (D) that are written out below.

(A)(*Kronecker products*) Look at the definition of Kronecker product in the Matrix Algebra appendix (Section A.2.5, p. 259) of MKB. Suppose that A, B are each symmetric nonnegative-definite $p \times p$ matrices with respective eigenvalues λ_i^A, λ_i^B and respective $p \times p$ orthogonal matrices V_A, V_B whose columns are orthonormal bases $\{V_A^{(i)}\}, \{V_B^{(i)}\}$ of eigenvectors in $\mathbb{R}^p$, such that

$$A\, V_A^{(i)} = \lambda_i^A\, V_A^{(i)} \quad , \qquad B\, V_B^{(i)} = \lambda_i^B\, V_B^{(i)}$$

(i). Show that $V_A \otimes V_B$ is a $p^2 \times p^2$ matrix of orthonormal eigenvectors of $A \otimes B$ with respective eigenvalues $\{\lambda_i^B \cdot \lambda_j^A\}_{(i,j)}$ with $1 \le i, j \le p$ and (i, j) ordered lexicographically.

(ii). Use (i) to show that $tr(A \otimes B) \equiv \sum_{(i,j)} \lambda_i^B \cdot \lambda_j^A = tr(B) \cdot tr(A)$ and $det(A \otimes B) \equiv \prod_{(i,j)} (\lambda_i^B \cdot \lambda_j^A) = (det(B) \cdot det(A))^p$.

(B) Suppose that $\mathbf{Y} = (Y_{ij})_{i,j=1,\ldots,p}$ is a $p \times p$ random matrix (alternately viewed as a random vector in $\mathbb{R}^{p^2}$) with a density $f(\mathbf{y})$ on $\mathbb{R}^{p^2}$.

(i) Show that Y is nonsingular with probability 1. (*Hint: use determinants.*)

(ii) Assuming that a matrix $A = A(t)$ is invertible and a differentiable function of t, use the identity $\frac{d}{dt}(A(t)^{-1}A(t)) = 0$ to prove that $\frac{d}{dt}A^{-1} = -A^{-1}\,\frac{dA}{dt}\,A^{-1}$.

After an email indicating a corrected statement, here is the final corrected form of the last part of the problem:

(iii) Now consider the inverse $\mathbf{Y}^{-1}$ as a p^2-dimensional vector-valued function of a p^2-dimensional vector, and consider its Jacobian as a $p^2 \times p^2$ matrix indexed by pairs $(i, j), (k, l)$ in $\{1, \ldots, p\}^2$. Use (ii) to conclude that the Jacobian of $\mathbf{Y}^{-1}$ viewed as a function of $\mathbf{Y}$ is the Kronecker product $-(\mathbf{Y}^{-1})^{tr} \otimes \mathbf{Y}^{-1}$, where $\mathbf{Y}^{-1}$ in this last Kronecker-product expression is the $p \times p$ matrix.

It follows from these two problems that the absolute Jacobian determinant of $\mathbf{Y}^{-1}$ as a function of the variables $\mathbf{Y}$ is $|det(\mathbf{Y})|^{-2p}$ whenever $\mathbf{Y}$ is invertible. This proves the first-row assertion of Table 2.4.1 on MKB page 36.

Continued on next page.

(C) Apply the Random Projection plotting technique (either my function `Ran.Proj` from the class demonstration and `R` script or your own code doing a similar thing) to the "thyroid" data that are included in the "mclust" `R`-package. Those data are in a 215×6 data-frame, the first column of which is a Diagnosis of "normal" or "Hyper" (-thyroid) or "Hypo", and the remaining 5 columns are medical measurements on the thyroid patients. Use the Random Projections to try to separate the Hyper and Hypo subjects (respectively, the records numbered 151:185 and 186:215) from the others. In your repeated Random Projections, keep track (from the visual output) of how often (the rough proportion of the time) the random projections effectively separate the indicated classes of subjects. By looking at the retained projection directions, find a way to summarize what you find out about which variables are most important in achieving discrimination between the different classes of subjects.

(D) Read in the dataset from Johnson and Wichern Table 1-9 saved as `NationalTrackRecords.txt` in the `/data/` directory on the course web-page. This dataset collects national women's record times in 7 track events, by country, for 54 countries. Here one might imagine a grouping of countries by size, by wealth, by commitment to developing top track and field performers; and similarly, groupings of events by similarity, especially the shorter-distance events versus the longer ones. Apart from the countries that are consistently top performers in all events, or that are less competitive in all events, we can ask whether there are countries that are very competitive in at least one but not all events, and whether there seems to be a tendency for countries to "specialize" in the sense that the variation of competitiveness across events is large but the performance in just a few similar events is consistently good. (My motivation in this problem is the small piece of knowledge that the Kenyan men and women are consistently great marathon runners but you never hear of them as sprinters.)

Find a way to display these data in such a way that you can help to answer this question (not formally, not definitively, but in one or more interesting pictures or tables). Since the times in different events are not comparable, I suggest [instead of the Johnson and Wichern book's recommendation to scale them by variance] that you rank the times in each type of event using the `rank` function in base-`R`, and that you also look at the variability (say, standard deviations) of these ranks within country across event.