THE MODULUS OF CONTINUITY FOR $\Gamma_0(m)\backslash\mathbb{H}$ SEMI-CLASSICAL LIMITS

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ABSTRACT. We study the behavior of a large-eigenvalue limit of eigenfunctions for the hyperbolic Laplacian for the modular quotient $SL(2;\mathbb{Z})\backslash\mathbb{H}$.Féjer summation and results of S. Zeldicth are used to show that the microlocal lifts of eigenfunctions have large-eigenvalue limit a geodesic flow invariant measure for the modular unit cotangent bundle. The limit is studied for Hecke-Maass forms, joint eigenfunctions of the Hecke operators and the hyperbolic Laplacian. The first modulus of continuity result is presented for the limit. The singular concentration set of the limit cannot be a compact union of closed geodesics and measured geodesic laminations.

1. INTRODUCTION

Let Δ be the Laplace-Beltrami operator for a finite volume Riemannian manifold M. The large-eigenvalue limit of eigenfunctions of Δ presents a model for the transition between quantum and classical mechanics [8]. The operator $e^{it\sqrt{-\Delta}}$ represents time evolution for the quantum mechanical system; geodesic flow represents time evolution for the classical mechanical system. In the large-eigenvalue limit the eigenfunctions (quantum states) give rise to a geodesic flow invariant measure (a classical state) on the unit cotangent bundle of M. The quantum ergodicity question is to understand the limit in the presence of a classical ergodic flow [2, 3, 4, 7, 16, 19, 20, 30, 31, 33]. The limit for finite area quotients of the hyperbolic plane and in particular modular quotients presents a setting where an explicit understanding is developing [11, 10, 13, 16, 19, 20, 21, 22, 28, 29, 30, 31, 32, 33, 34]. The quantum ergodicity question for hyperbolic quotients involves modular functions, coefficient sums and the structure of $SL(2;\mathbb{R})$.

A basic construction is the microlocal lift of a Laplace-Beltrami eigenfunction. The lift is an almost measure (a distribution) on the unit

Date: March 13, 2000.

1991 *Mathematics Subject Classification.* Primary 11F72, 58J40; Secondary 11F03, 30F35.

Research supported in part by NSF Grant DMS-9800701.

cotangent bundle $S^*(M)$; the first term of the lift is the eigenfunction square. For large eigenvalue the lift is almost invariant under geodesic flow. A. Schnirleman [22], Y. Colin de Verdière [7], and S. Zelditch [31] showed for compact manifolds with ergodic geodesic flow that the spectral average of the microlocal lifts is the uniform distribution on $S^*(M)$; a corollary provides for a full spectral density subsequence that the microlocal lifts converge to the uniform density on $S^*(M)$. S. Zelditch first considered non compact hyperbolic quotients. The corresponding spectral decomposition for the hyperbolic Laplacian consists of the continuous span of the Eisenstein series and the span of the square integrable eigenfunctions, [5, 23, 27]. S. Zelditch found the appropriate renormalization for the Eisenstein series and showed that the spectral average again is the uniform distribution, [34]. For $SL(2;\mathbb{Z})$ the Eisenstein series contribution in fact has smaller order of magnitude and does not contribute to the spectral average, [34]. W. Luo and P. Sarnak were able to directly analyze the modular Eisenstein series [16]. They found that the absolute square of the Eisenstein series weak* converges to $48\pi^{-1}$ for large-eigenvalues; their analysis involved the subconvexity bounds for the Riemann zeta function and the L-functions for Maass cusp forms. D. Jakobson extended the considerations to include the microlocal lift of the Eisenstein series [13]. In [29] the author found that the microlocal lift to $SL(2;\mathbb{R}) \approx S^*(\mathbb{H})^{1/2}$ of automorphic eigenfunctions can be obtained directly from their twisted Fourier coefficient sums. The Luo-Sarnak and Jakobson result is equivalent to a limit-sum formula combining the Riemann zeta values $\zeta(1+it)$ and the elementary divisor values σ_{2it}.

Z. Rudnick and P. Sarnak considered arithmetic compact hyperbolic quotients [19]. An Eichler order in a quaternion algebra over $\mathbb{Q}$ gives rise to a cocompact subgroup $\Gamma \subset SL(2;\mathbb{R})$ with a commutative ring of self-adjoint operators, Hecke operators, acting on $L^2(\Gamma\backslash\mathbb{H})$ and commuting with the hyperbolic Laplacian. Closed geodesics for such a Γ are associated with binary quadratic forms. There is a computational scheme for determining the action of the Hecke operators on closed geodesics. The authors show that a closed geodesic can be separated from any finite set of closed geodesics by a Hecke operator. The result provides for joint eigenfunctions of the hyperbolic Laplacian and the Hecke operators that a large-eigenvalue limit cannot have singular support a finite union of points and closed geodesics, [19, Theorem 1.1].

S. Zelditch introduced a microlocal lift to $SL(2;\mathbb{R})$ based on Helgason's Fourier transform [12, 33]. He found that the lift satisfies an exact differential equation; see Lemma 2 below. Properties of the large-eigenvalue limit of $SL(2;\mathbb{R})$ microlocal lifts can be obtained directly from Fejér summation and integration by parts: see Proposition 4 for the basic properties and Proposition 5 for Cauchy-Schwartz and Minkowski type inequalities. We consider the Hecke operators for $SL(2;\mathbb{Z})$ and the congruence subgroups $\Gamma_0(m)$. We describe a sub-tiling for the Hecke operators T_p, $p \leq q$ and a basic set of diameter q^{-2}. We combine the sub-tiling for the Hecke operators, the structure of the microlocal lift and the partial-sums for $\sum p^{-1}$ to study limits of the lifts. The measure of a set is estimated after tiling a region with translates of the set. We find in particular for joint eigenfunctions of the hyperbolic Laplacian and the Hecke operators that a large-eigenvalue limit of microlocal lifts with compact singular support vanishes on each closed geodesic and on each geodesic lamination for a finite index subgroup. Our results and approach have similarities to the work of D. Jakobson and S. Zelditch on semi-classical limits for eigenfunctions of Hecke operators for the sphere $\mathbb{S}^2$ [**?**, Section 4.3]. In comparison to the considerations of Z. Rudnick and P. Sarnak the present result provides that even more general limit measures will be null on closed geodesics and geodesic laminations. A limit measure is geodesic flow invariant and hence determines a measure on the leaf space for the flow, the space of geodesics for the hyperbolic plane. We present the first explicit *modulus of continuity* bound for such measures. The mass in a ball of radius ϵ is bounded by $(\log\log\epsilon^{-1})^{-2}$; see Proposition 10 below.

2. Background

We recall the formalism for $SL(2;\mathbb{R})$, [15], as well as the construction of S. Zelditch for the microlocal lift [30, 31, 33]. An element $B \in SL(2;\mathbb{R})$ has the unique Iwasawa decomposition

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

which provides for an equivalence of $SL(2;\mathbb{R})$ with $S^*(\mathbb{H})^{1/2}$, the square root of the unit cotangent bundle to the upper half plane, by the rule

$$x + iy = y^{1/2}e^{i\theta}(ai+b), \quad y^{-1/2}e^{i\theta} = d - ic$$

for $z = x + iy \in \mathbb{H}$ and θ the argument for the root cotangent vector measured from the positive vertical. The equivalence will play a basic role throughout. The bi-invariant volume form (Haar measure) for $SL(2;\mathbb{R})$ is $d\mathcal{V} = y^{-2}dxdyd\theta$. The Lie algebra acts on the right of

$SL(2;\mathbb{R})$ with $E^{\pm} = H \pm iV$ for

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ W = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ \text{and } X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The infinitesimal generator of geodesic-flow is $H = \frac{1}{2}(E^+ + E^-)$; W is the infinitesimal generator of K, the fiber rotations of $S^*(\mathbb{H})^{1/2}$. In terms of the coordinates (x, y, θ) for $SL(2;\mathbb{R})$ the operator E^+ is simply $E^+ = 4iye^{2i\theta}\frac{\partial}{\partial z} - ie^{2i\theta}\frac{\partial}{\partial\theta}$ and the operator X is simply $y\cos 2\theta\frac{\partial}{\partial x} + y\sin 2\theta\frac{\partial}{\partial y} + y\sin^2\theta\frac{\partial}{\partial\theta}$, [15].

A function u on $\mathbb{H}$ satisfying the differential equation $Du + (\frac{1}{4} + r^2)u = 0$, D the hyperbolic Laplacian, lifts to a K-invariant function on $SL(2;\mathbb{R})$ satisfying $\mathcal{C}u = (2ir+1)(2ir-1)u$ for the Casimir operator $\mathcal{C} = E^-E^+ - W^2 - 2iW$. The Casimir operator is in the center of the enveloping algebra. A ladder of functions, the *raisings* and *lowerings* of u, is determined by the scheme

$$\begin{aligned} u_0 &= u \\ (2ir + 2m + 1)u_{2m+2} &= E^+u_{2m} \\ (2ir - 2m + 1)u_{2m-2} &= E^-u_{2m} \end{aligned} \tag{1}$$

for m integral. The function u_{2m} is in the weight $2m$ irreducible representation for K as demonstrated by $Wu_{2m} = i2m\ u_{2m}$. The sum $u^\infty = \sum_m u_{2m}$ is a distribution that is N-invariant as well as an eigendistribution of H [31, Prop. 2.2][33, pg.44;].

Elements of the Lie algebra $sl(2;\mathbb{R})$ preserve the volume form and can be integrated by parts. In particular the integral $\int_{\mathcal{Q}} B\kappa d\mathcal{V}$ vanishes for B in the Lie algebra and $\mathcal{Q} = SL(2;\mathbb{R})$ or $\mathcal{Q} = \Gamma\backslash SL(2;\mathbb{R})$, Γ a discrete subgroup, with κ a smooth compactly $\mathcal{Q}$-supported function. Consider solutions u, v of the equation $\mathcal{C}u = (2ir+1)(2ir-1)u$ and a smooth function χ. Provided χ is smooth with compact support there is the relation

$$0 = \int_{\mathcal{Q}} ((E^+u_{2j})\overline{v_{2k}}\chi + u_{2j}\overline{E^-v_{2k}})\chi + u_{2j}\overline{v_{2k}}E^+\chi d\mathcal{V}. \tag{2}$$

We are ready to consider the microlocal lifts of automorphic eigenfunctions. Let $\Gamma \subset SL(2;\mathbb{R})$ be a cofinite subgroup and φ an $L^2(\Gamma\backslash\mathbb{H})$ eigenfunction with unit-norm. The function φ lifts to a K-invariant function on $SL(2;\mathbb{R})$ satisfying $\mathcal{C}\varphi = (2ir+1)(2ir-1)\varphi$. We consider the ladder $\{\varphi_{2m}\}$ of *raisings* and *lowerings*, as well as the quantity $\varphi^\infty = \sum_m \varphi_{2m}$. The ladder $\{\varphi_{2m}\}$ is an orthogonal basis for an irreducible *principal continuous series* representation of $SL(2;\mathbb{R})$, [15].

For the $L^2(\Gamma\backslash SL(2;\mathbb{R}))$ Hermitian product $\langle\varphi_{2m},\varphi_{2m}\rangle = 2\pi$ is satisfied and from integration by parts $\langle E^+\varphi_{2j},\varphi_{2k}\chi\rangle + \langle\varphi_{2j},E^-(\varphi_{2k}\chi)\rangle = 0$ for a Γ-invariant test function χ.

A test function $\chi \in C_c^2(\Gamma\backslash SL(2;\mathbb{R}))$ has a K Fourier expansion $\chi = \sum_m \chi_m$ with $|\chi_m| \leq C_\chi(1+|m|)^{-2}$. For $L^2(\Gamma\backslash\mathbb{H})$ eigenfunctions φ and ψ from Parseval's relation the pairing of χ with $\varphi\overline{\psi^\infty}$ is the sum

$$\sum_m \int_{\Gamma\backslash SL(2;\mathbb{R})} \chi_{2m}\varphi\overline{\psi_{2m}}d\mathcal{V}.$$

The sum is bounded by $C_\chi\|\varphi\|\|\psi\|$. In consequence the quantity $\varphi\overline{\psi^\infty}$ is a distribution for $C_c^2(\Gamma\backslash SL(2;\mathbb{R}))$. Equivalently $\varphi\overline{\psi^\infty}$ is a distribution for $C^2_{c,\Gamma}(SL(2;\mathbb{R}))$, the $C_c^2(SL(2;\mathbb{R}))$ subspace of Γ-invariant functions. Relatedly the operator $\Theta : C_c^2(SL(2;\mathbb{R})) \rightarrow C_c^2(\Gamma\backslash SL(2;\mathbb{R}))$ defined by $\Theta\chi = \sum_{\gamma\in\Gamma}\chi \mid \gamma$ is a continuous surjection relative to the Fréchet topologies; furthermore for functions with support contained in a Γ-fundamental domain $\|\chi\| = \|\Theta\chi\|$. In consequence a distribution for $C_c^2(\Gamma\backslash SL(2;\mathbb{R}))$ has a natural extension, the formal adjoint of Θ, to a distribution for $C_c^2(SL(2;\mathbb{R}))$; furthermore convergence of extensions is equivalent to convergence of the original distributions. In the following considerations we will use all three settings for the distribution $\varphi\overline{\psi^\infty}$.

Definition 1. *For $L^2(\Gamma\backslash\mathbb{H})$ eigenfunctions φ and ψ set $2\,Q(\varphi,\psi) = \varphi\overline{\psi^\infty} + \psi\overline{\varphi^\infty}$ and $Q(\varphi) = Q(\varphi,\varphi)$.*

The microlocal lift $Q(\varphi)$ is a basic quantity for the ΨDO-calculus based on Helgason's Fourier transform [12, 33]. For $\sigma \in C^\infty(SL(2;\mathbb{R})\times\mathbb{R})$, a complete symbol for a ΨDO compactly supported on $SL(2;\mathbb{R})$ ($\sigma(A,\tau)$ is asymptotically a sum of homogeneous terms in the frequency τ with bounded left invariant derivatives in A) the associated matrix element is

$$2\pi\langle Op(\sigma)v,u\rangle = \int_{SL(2;\mathbb{R})} \sigma_r\overline{Q(u,v)}d\mathcal{V}$$

for σ_r the symbol evaluated at $\tau = r$ and $-\frac{1}{4} - r^2$ the eigenvalue for u, v [30, 33]. S. Zelditch discovered that the essential properties of the microlocal lift are given by an exact differential equation [33]

Lemma 2. *For φ and ψ weight zero eigenfunctions of the Casimir operator with eigenvalue $-(4r^2+1)$ then $(H^2+4X^2+4irH)Q(\varphi,\psi) = 0$.*

A Lie algebraic proof of the Lemma is presented in [29].

We are interested in the large-eigenvalue limit of a sequence of automorphic eigenfunctions for Γ a cofinite subgroup. As noted above for a sequence $\{\varphi_n\}$ of $L^2(\Gamma\backslash\mathbb{H})$ unit-norm eigenfunctions the sequence $\{Q(\varphi)\}$ of $C^2_c(SL(2;\mathbb{R}))$ (and thus $C^2_c(\Gamma\backslash SL(2;\mathbb{R}))$) distributions is precompact. Provided the eigenvalues tend to infinity then from Lemma 2 [33] the limit of a convergent subsequence is a geodesic flow invariant distribution for $C^2_c(SL(2;\mathbb{R}))$.

Definition 3. *A sequence of normalized $L^2(\Gamma\backslash\mathbb{H})$ real-valued eigenfunctions $\{\varphi_n\}$ with eigenvalues tending to infinity has semi-classical limit μ_φ provided $\mu_\varphi = \lim_n Q(\varphi_n)$ in the sense of $C^2_c(SL(2;\mathbb{R}))$ distributions.*

We now consider an alternate construction for the microlocal lift in terms of Fejér summation of the ladder of $SL(2;\mathbb{R})$ raisings and lowerings, [29]. For an eigenfunction φ and a positive integer M introduce the sum

$$Q_M(\varphi) = (2M+1)^{-1}\Big|\sum_{m=-M}^{M}\varphi_{4m}\Big|^2.$$

The basic properties of the semi-classical limit are in fact simple consequences of the Fejér summation and integration by parts.

Proposition 4. *Notation as above. Let $\{\varphi_n\}$ be a sequence with semi-classical limit μ_φ . The limit satisfies $\mu_\varphi = \lim_M \lim_n Q_M(\varphi_n)$, is a positive real measure on $SL(2;\mathbb{R})$ and is time-reversal invariant (right $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ invariant). Let $\{(\varphi_n,\psi_n)\}$ be a sequence of pairs of eigenfunctions, φ_n and ψ_n with common eigenvalue, eigenvalues tending to infinity, such that $\{Q(\varphi_n)\},\{Q(\psi_n)\}$ and $\{Q(\varphi_n,\psi_n)\}$ converge in the sense of $C^2_c(SL(2;\mathbb{R}))$ distributions. The limit $\lim_n Q(\varphi_n,\psi_n)$ is a real time-reversal invariant measure on $SL(2;\mathbb{R})$.*

Proof. A semi-classical limit is determined on the subspace C^∞_c of C^2_c; furthermore the K expansions are convergent for a convergent sequence of distributions. First we show that the terms $\varphi\overline{\varphi_{2m}}$ with m odd do not contribute to the limit. From (1) we have that $\overline{u_{-2m}} = (-1)^m(\bar{u})_{2m} + O(r^{-1}|u_{2m}|)$ in the sense of distributions. Since φ is real we have that $\varphi\overline{\varphi_{-2m}} = -\varphi\varphi_{2m} + O(r^{-1})$. Now by a repeated application of (2) we have for $m = 2q+1$ that $4i\varphi\varphi_{2m} = (-1)^q r^{-1}E^+((\varphi_{2q})^2) + O(r^{-1})$. The leading-term $E^+((\varphi_{2q})^2)$ is itself a bounded distribution; in consequence for m odd $\varphi\overline{\varphi_{2m}}$ and $\varphi\overline{\varphi_{-2m}}$ have magnitude $O(r^{-1})$ and thus do not contribute to a limit. In particular the limits μ_φ and $\mu_{\varphi+\psi}$ have K expansions with non trivial terms only for weights congruent to zero modulo 4; the limits are time-reversal invariant. From (2) we have the additional relation $\varphi_{2j+2}\overline{\psi_{2k}} = \varphi_{2j}\overline{\psi_{2k-2}} + O(r^{-1})$ in the sense of distributions. It follows for pairs of eigenfunctions that $\lim_n Q(\varphi_n,\psi_n) = \lim_n \overline{Q(\psi_n,\varphi_n)}$ and consequently that a limit is real. It further follows that $\lim_n Q_M(\varphi_n) = \lim_n \sum_{m=-2M}^{2M}(1-\frac{|m|}{2M+1})\varphi_n\overline{\varphi_{n,4m}}$ and from the above result on the K expansion that $\lim_M \lim_n Q_M(\varphi_n) = \mu_\varphi$. Finally since Q_M is positive it follows that μ_φ is a positive real measure. The proof is complete.

We consider further properties for the semi-classical limit of tuples of eigenfunctions. Consider a sequence $\{(\varphi_n,\psi_n)\}$ of pairs of eigenfunctions, φ_n and ψ_n with common eigenvalue λ_n, such that for eigenvalues tending to infinity the microlocal lifts converge to measures on $SL(2;\mathbb{R})$

$$\mu_\varphi = \lim_n Q(\varphi_n),\ \mu_\psi = \lim_n Q(\psi_n),\ \mu_{\varphi\pm\psi} = \lim_n Q(\varphi_n\pm\psi_n) \text{ and } \mu_{\varphi,\psi} = \lim_n Q(\varphi_n,\psi_n).$$

Proposition 5. *Notation as above. The measures satisfy* $2|\mu_{\varphi,\psi}| \leq \mu_\varphi+\mu_\psi$ *and* $(\int\chi\mu_{\varphi,\psi})^2 \leq \int\chi\mu_\varphi\int\chi\mu_\psi$ *for each positive* $\chi\in C_c(SL(2;\mathbb{R}))$. *In particular* $\mu_{\varphi,\psi}$ *is absolutely continuous with respect to* μ_φ *and to* μ_ψ.

Proof. The pair of inequalities $\pm 2\mu_{\varphi,\psi} \leq \mu_\varphi+\mu_\psi$ are a consequence of the positivity of the measures $\mu_{\varphi\pm\psi}$. The first assertion is now a consequence of the Jordan decomposition of $\mu_{\varphi,\psi}$ as a difference of mutually singular positive measures [18]. For the second assertion consider a non negative test function χ and the quadratic polynomial $\int\chi(\alpha^2\mu_\varphi+2\alpha\mu_{\varphi,\psi}+\mu_\psi)$ in the real parameter α. Since $\mu_{\alpha\varphi+\psi}$ is a positive measure for each α the second assertion now follows. The measures μ_* are outer regular: for a compact Borel set $\mathcal{S}$ then $\mu_*(\mathcal{S}) = \inf_\chi \int \chi\mu_*$ for $\chi\in C_c(SL(2;\mathbb{R}))$ with $\chi=1$ on $\mathcal{S}$, [18]. In particular for compact Borel sets we find $(\mu_{\varphi,\psi}(\mathcal{S}))^2 \leq \mu_\varphi(\mathcal{S})\mu_\psi(\mathcal{S})$ and thus that $\mu_{\varphi,\psi}$ is absolutely continuous with respect to μ_φ and μ_ψ. The proof is complete.

Corollary 6. *Notation as above. For a sequence of q-tuples of eigenfunctions* $\{(\varphi_{1,n},\dots,\varphi_{q,n})\}$, $\varphi_{j,n}$ *with common eigenvalues, and all pairs* $Q(\varphi_{j,n},\varphi_{k,n})$ *convergent for eigenvalues tending to infinity then* $\mu_\Phi \leq q(\sum_{j=1}^q \mu_{\varphi_j})$ *for* $\Phi = \sum_{j=1}^q \varphi_j$.

Proof. The positive measure μ_Φ is given as a sum $\mu_\Phi = \sum_{j=1}^q \mu_{\varphi_j} + 2\sum_{1\leq j<k\leq q}\mu_{\varphi_j,\varphi_k}$. The result follows from the inequality $2|\mu_{\varphi,\psi}| \leq \mu_\varphi + \mu_\psi$. The proof is complete.

3. Modular limits

We wish to investigate semi-classical limits for the congruence subgroups $\Gamma_0(m) = \{(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in SL(2;\mathbb{Z}) \mid c \equiv \bmod m\}$; $\Gamma_0(1) = SL(2;\mathbb{Z})$. The Hecke operators T_p, p a prime, $p \nmid m$ act on $L^2(\Gamma_0(m)\backslash SL(2;\mathbb{R}))$; the operators are self-adjoint, mutually commuting and commute with the Casimir operator as well as geodesic flow [1, 23, 27]. The Hecke operators are defined from a left action on $SL(2;\mathbb{R})$ and so necessarily commute with the right action of the Lie algebra. For φ a $\Gamma_0(m)$-invariant function on $\mathbb{H}$ and $p \nmid m$ then

$$\varphi \mid T_p = \sum_{j=0}^{p-1} \varphi \mid A_p^{-1} S^j + \varphi \mid A_p$$

for $A_p = \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $S = (\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix})$. Since T_p commutes with the action of the Lie algebra it follows that $(\varphi \mid T_p)^\infty = \varphi^\infty \mid T_p$ for an eigenfunction φ on $\mathbb{H}$. The spectral decomposition for the hyperbolic Laplacian acting in $L^2(\Gamma_0\backslash\mathbb{H})$ consists of the continuous span of the Eisenstein series and $L^2_{int}(\Gamma_0\backslash\mathbb{H})$ the subspace spanned by square-integrable eigenfunctions, [5, 23, 27]. The family $\{D, T_p\}$ consists of mutually commuting operators on $L^2_{int}(\Gamma_0\backslash\mathbb{H})$. Mutual eigenfunctions of $\{D, T_p\}$ are referred to as *Hecke eigenforms.*

Consider now $\{\psi_n\}$ a sequence of Hecke eigenforms with semi-classical limit μ_ψ. The measure μ_ψ is invariant under geodesic flow and consequently is a linear combination of Haar measure $d\mathcal{V}$ and a totally singular measure $\operatorname{sing}(\mu_\psi)$. The Hecke operator eigenequations give rise to relations for the measure μ_ψ.

Proposition 7. *Notation as above. Let $\{\psi_n\}$ be a sequence of Hecke eigenforms with semi-classical limit μ_ψ satisfying $\operatorname{sing}(\mu_\psi)$ has compact support in $\Gamma_0(m)\backslash SL(2;\mathbb{R})$. Given a compact set $\mathcal{K} \subset SL(2;\mathbb{R})$ for all sufficiently large primes $\mu_\psi(\mathcal{B}) \leq p \sum_{j=1}^{p-1} \mu_\psi(S^{j/p}\mathcal{B})$ for $S = (\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix})$ and each Borel set $\mathcal{B} \subset \mathcal{K}$.*

Proof. We begin with the formula $A_p^{-1} S^j A_p = \left(\begin{smallmatrix} 1 & j/p \\ 0 & 1 \end{smallmatrix}\right) = S^{j/p}$ and in consequence for a Hecke eigenform ψ for $p \nmid m$ with $\psi \mid T_p = \alpha_p \psi$ we have the equation

$$\psi = \alpha_p \psi \mid A_p - \psi \mid A_p^2 - \sum_{j=1}^{p-1} \psi \mid S^{j/p}.$$

Given $\mathcal{K}$ compact in $SL(2;\mathbb{R})$ for all sufficiently large primes $A_p(\mathcal{K})$ is disjoint from the support $\operatorname{supp}(\sigma_\psi)$, $\sigma_\psi = \operatorname{sing}(\mu_\psi)$, (this is apparent on considering the projection from $SL(2;\mathbb{R})$ to $\mathbb{H}$). In consequence for p large $\mu_\psi \mid A_p$ is absolutely continuous with respect to Lebesgue measure and $\sigma_\psi(A_p(\mathcal{K})) = 0$. Now from Proposition 5 the measures $\lim_n Q(\alpha_{p,n}\psi_n \mid A_p - \psi_n \mid A_p^2)$ and $\lim_n Q(\alpha_{p,n}\psi_n \mid A_p - \psi_n \mid A_p^2, \sum_{j=1}^{p-1}\psi_n \mid S^{j/p})$ are absolutely continuous with respect to $\mu_\psi \mid A_p = \lim_n Q(\psi_n \mid A_p)$ (the eigenvalues $\alpha_{p,n}$ are all bounded by $p+1$ from the elementary bound of E. Hecke [23]). We thus deduce the equality of totally singular measures $\mathrm{sing}(\mu_\psi) = \mathrm{sing}(\lim_n Q(\sum_{j=1}^{p-1}\psi_n \mid S^{j/p}))$ on $\mathcal{K}$. The inequality for $\mathrm{sing}(\mu_\psi)$ now follows from Corollary 6. The semi-classical limit μ_ψ is a linear combination of $\mathrm{sing}(\mu_\psi)$ and Haar measure; Haar measure is $SL(2;\mathbb{R})$ invariant and trivially satisfies the stated inequality. The linear combination satisfies the inequality. The proof is complete.

We now wish to reformulate the above result for the space of complete geodesics on the upper half plane. The reformulation will enabale a later argument. Geodesic flow provides a fibration by trajectories $SL(2;\mathbb{R}) \to SL(2;\mathbb{R})/\{e^{tH} \mid t \in \mathbb{R}\}$. A geodesic on $\mathbb{H}$ has two unit cotangent fields and four unit square-root cotangent fields; $SL(2;\mathbb{R})/\{e^{tH} \mid t \in \mathbb{R}\}$ is a four-fold cover of $\mathbb{G}$ the space of geodesics. The four-fold covering provides a (left $SL(2;\mathbb{R})$ action) natural correspondence for measures. In particular a measure κ on $\mathbb{G}$ corresponds to the geodesic flow invariant, right $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ invariant, measure $\widetilde{\kappa}d\ell$ on $SL(2;\mathbb{R})$ (for $\widetilde{\kappa}$ the lift of κ to $SL(2;\mathbb{R})/\{e^{tH} \mid t \in \mathbb{R}\}$ and $d\ell$ the infinitesimal flow time). The naturality property is the relation $\widetilde{\kappa \mid B}d\ell = (\widetilde{\kappa}d\ell) \mid B$ for each $B \in SL(2;\mathbb{R})$. The Hecke operators on $L^2(\Gamma_0(m)\backslash SL(2;\mathbb{R}))$ are defined from a left action on $SL(2;\mathbb{R})$ and thus we can reformulate the above Proposition. From the discussion of the prior section the semi-classical limit can be considered as a positive measure on $SL(2;\mathbb{R})$ and thus can just as readily be bounded in terms of non Γ-invariant quantities.

Corollary 8. *Notation as above. Let $\{\psi_n\}$ be a sequence of Hecke eigenforms with semi-classical limit μ_ψ having compact singular support in $\Gamma_0(m)\backslash SL(2;\mathbb{R})$. For τ_ψ the corresponding measure on $\mathbb{G}$ and $\mathcal{K} \subset \mathbb{G}$ compact for all sufficiently large primes $\tau_\psi(\mathcal{B}) \le p\sum_{j=1}^{p-1}\tau_\psi(S^{j/p}\mathcal{B})$ for each Borel set $\mathcal{B} \subset \mathcal{K}$.*

Proof. We choose a continuous section σ for the projection $SL(2;\mathbb{R}) \to \mathcal{T} = SL(2;\mathbb{R})/\{e^{tH} \mid t \in \mathbb{R}\}$. The section provides a lifting of points $\beta \in \mathcal{T}$ to intervals $I(\beta) = \{\sigma(\beta)e^{tH} \mid 0 \le t \le 1\}$; compact sets are lifted to compact sets and for a measure ν on $\mathbb{G}$ and Borel set $\mathcal{B} \subset \mathbb{G}$, $\widetilde{\mathcal{B}}$ the lift to $\mathcal{T}$, then $4\,\nu(\mathcal{B}) = (\widetilde{\nu}d\ell)(I(\widetilde{\mathcal{B}}))$. The desired result now follows from the previous proposition. The proof is complete.

We wish to consider the consequences of the above Corollary for the measure τ_ψ. For this purpose consider the fibration $\Gamma_{\mathbb{Z}}\backslash\mathbb{G} \rightarrow \mathcal{H} = \{S^t = \left(\begin{smallmatrix}1 & t\\ 0 & 1\end{smallmatrix}\right) \mid t \in \mathbb{R}\}\backslash\mathbb{G}$, $\Gamma_{\mathbb{Z}}$ the group of integer translations. We say that a Borel set $\mathcal{B} \subset \Gamma_{\mathbb{Z}}\backslash\mathbb{G}$ is *height determined* provided the projection to $\mathcal{H}$ restricted to $\mathcal{B}$ is an injection (in particular $\Gamma_{\mathbb{Z}}\backslash\mathcal{B}$ contains at most one geodesic on $\Gamma_{\mathbb{Z}}\backslash\mathbb{H}$ of each height). For an interval $I \subset \mathbb{R}$ and the height determined set $\mathcal{B}$ we will consider the thickened set $\mathcal{B}_I = \cup_{t\in I} S^t(\mathcal{B}) \subset \Gamma_{\mathbb{Z}}\backslash\mathbb{G}$. We now combine the above inequality and the observation that the parabolics $S^{j/p}, 1 \leq j \leq p-1, p \leq q$ give a *sub-tiling* of the set $\mathcal{B}_{(0,1)}$ by the basic set $\mathcal{B}_{(-q^{-2},\,q^{-2})}$. The result is an explicit bound for the mass of the basic set.

Proposition 9. *Notation as above. For τ_ψ as above given a compact set $\mathcal{K} \subset \Gamma_{\mathbb{Z}}\backslash\mathbb{G}$ there exists a positive constant C such that for a height determined set $\mathcal{B} \subset \mathcal{K}$ then $\tau_\psi(\mathcal{B}_{(-\epsilon,\epsilon)}) \leq C(\log\log\epsilon^{-1})^{-1}$ for all $\epsilon < e^{-1}$. In particular a height determinded set is a null set for τ_ψ.*

Proof. It suffices given $\mathcal{K}$ to provide a bound for all small ϵ. For q the positive integer satisfying $(q+1)^{-2} < \epsilon \leq q^{-2}$ and $\mathcal{B}$ a height determined set, consider the set $\mathcal{B}_q = \mathcal{B}_{(-q^{-2},\,q^{-2})}$. We have the inclusion $\mathcal{B}_{(-\epsilon,\,\epsilon)} \subset \mathcal{B}_q$, as well as disjointness of $S^{j/p}\mathcal{B}_q$ from $S^{k/p'}\mathcal{B}_q$ for all $p < p' \leq q/2$ (since $|j/p - k/p'| \geq (pp')^{-1} \geq 4q^{-2}$ and q^{-2}-neighborhoods of j/p, k/p' are disjoint).

There are consequences for the values $\tau_\psi(\mathcal{B}_*)$. For p_0 the threshold for the conclusion of Corollary 8 for a compact set $\mathcal{K}$ we have the inequality

$$\sum_{p=p_0}^{q} p^{-1}\tau_\psi(\mathcal{B}_q) \leq \sum_{p=p_0}^{q}\sum_{j=1}^{p-1}\tau_\psi(S^{j/p}\mathcal{B}_q).$$

The left hand side is immediately bounded below by the product of $\tau_\psi(\mathcal{B}_{(-\epsilon,\epsilon)})$ and $\sum_{p=p_0}^{q} p^{-1} \geq c\log\log q^{-1}$ for a positive constant [9]. For the right hand side we have that $S^{j/p}\mathcal{B}_q \subset \mathcal{B}_{(0,1)}$ for $p_0 \leq p \leq q/2,\ 1 \leq j \leq p-1$ and that the individual sets $S^{j/p}\mathcal{B}_q$ are mutually disjoint. It follows that the right hand side is bounded above by $\tau_\psi(\mathcal{B}_{(0,1)}) \leq \tau_\psi(\mathcal{K}_{(0,1)})$. The set $\mathcal{K}_{[0,1]}$ is compact and has finite τ_ψ measure. The proof is complete.

The previous result enables a modulus of continuity estimate for τ_ψ. Each cusp of Γ on $\mathbb{H}$ provides a one-parameter family of parabolics.

We use a pair of transverse families to first thicken a point of $\mathbb{G}$ to obtain an arc and to then thicken the arc to obtain a neighborhood. To formulate the result we fix a Riemannian metric for $\mathbb{G}$; a metric determines ϵ-neighborhoods $\mathbb{B}(*;\epsilon)$.

Proposition 10. *Notation as above. For τ_ψ as above given a compact set $\mathcal{K}\subset\mathbb{G}$ there is a positive constant C such that $\tau_\psi(\mathbb{B}(\gamma;\epsilon))\leq C(\log\log\epsilon^{-1})^{-2}$ for $\gamma\in\mathcal{K}$ and $\epsilon<e^{-1}$.*

Proof. The plan is to prescribe an ϵ-neighborhood by $SL(2;\mathbb{R})$ subgroup orbit segments. The considerations are local and thus $\mathbb{G}$ can be substituted in place of $\Gamma_{\mathbb{Z}}\backslash\mathbb{G}$. We first show that given $\gamma\in\mathbb{G}$ there exists a Γ-conjugate T of S such that the family $\{T^t\gamma\mid -\epsilon<t<\epsilon\}$ is a height determined set. We consider the projection of the family to $\mathcal{H}=\{S^t\}\backslash\mathbb{G}$; the families for T and $S^\delta TS^{-\delta}$ have the same projection. It suffices to consider a parabolic transformation fixing the origin and after a possible parameter rescaling to simply consider the special parabolic family $\{\left(\begin{smallmatrix}1&0\\t&1\end{smallmatrix}\right)\}$. We consider the action of the special family on the height of a geodesic γ. If γ has endpoints $a,b\in\hat{\mathbb{R}}$ with $a<b\leq\infty$ then the height of γ is $(b-a)/2$ (∞ if $b=\infty$). The first derivative of the height of $\{\left(\begin{smallmatrix}1&0\\t&1\end{smallmatrix}\right)\}\gamma$ at $t=0$ is $(a^2-b^2)/2$ (for $b=\infty$ the height function satisfies $height^{-1}=2t(1+at)$). It follows for ξ the fixed-point of T and $\xi<a$ that the height of $T^t\gamma$ is an injective function of t small; $T^t\gamma$ is height determined for $\xi<a$ for t small. Now the Γ-conjugates of S have fixed-points dense in $\mathbb{R}$ and we can select a conjugate $T=RSR^{-1}$, $R\in\Gamma$ with fixed point located as desired relative to γ.

We apply Proposition 9 for the singleton $\{\gamma\}$ (and R-conjugates) to conclude $\tau_\psi(\{\gamma\}_{(-\epsilon,\epsilon)})\leq C(\log\log\epsilon^{-1})^{-1}$ (the orbit segment relative to T is an arc). From the above paragraph the arc $\{\gamma\}_{(-\epsilon,\epsilon)}$ is height determined relative to S. We apply Proposition 9 a second time to obtain an ϵ-neighborhood of γ and the desired bound. The proof is complete.

Measured geodesic laminations furnish examples of locally height determined sets of geodesics, [6, 14, 24, 25]. A closed set in $\mathbb{H}$ is a *geodesic lamination* provided the set is a union of mutually disjoint complete (isometric to $\mathbb{R}$) geodesics. The individual geodesics of the union are the *leaves* of the lamination. A geodesic lamination $\mathcal{G}$ has a natural lift to the unit cotangent bundle and to $SL(2;\mathbb{R})$ since a point of $\mathcal{G}$ lies on a unique leaf which determines two unit cotangent vectors and four root cotangent vectors. Accordingly $\mathcal{G}$ determines a closed subset of the space of geodesics $\mathbb{G}$. We are interested in *measured geodesic laminations*: Γ-invariant geodesic laminations (with no leaves ending in a cusp) which considered in $\mathbb{G}$ are the full support of a Γ-invariant positive measure. A measured geodesic lamination determines a geodesic flow invariant measure on $SL(2;\mathbb{R})$ (a candidate for a semi-classical limit). For the sake of exposition we cite two basic results of W. Thurston. First for a surface of genus g with n punctures the space of measured geodesic laminations is parameterized by $\mathbb{R}^{6g-6+2n}$, [17, 26]. Second, the intersection of a measured geodesic lamination $\mathcal{G}$ and a transverse arc is the union of a finite set (supporting a sum of Dirac measures associated to closed simple geodesics on $\Gamma\backslash\mathbb{H}$) and a Cantor set (supporting a totally singular measure with no point masses) [6].

Proposition 11. *The support of a measured geodesic lamination for a cofinite group with a cusp at infinity is locally height determined.*

Proof. Consider a neighborhood in $\mathbb{G}$ of a non vertical geodesic γ. A translate $S^t\gamma$ is close to γ only if t is small in which case the endpoints of $S^t\gamma$ and γ alternate on $\mathbb{R}$ and consequently $S^t\gamma$ intersects γ. Since the leaves of a lamination are disjoint the conclusion follows. The proof is complete.

We wish to consider flow invariant sets for $\Gamma\backslash SL(2;\mathbb{R})$ more general than supports of measured geodesic laminations. The first are lifts of non simple closed geodesics. The second are the supports of Γ' measured geodesic laminations for $\Gamma' \subset \Gamma$ finite index subgroups. We are ready to present the main result.

Theorem 12. *A semi-classical limit for $\Gamma_0(m)$ with compact singular support is null on each countable union of closed geodesics and geodesic laminations for finite index subgroups.*

Proof. It suffices to consider individual closed geodesics and geodesic laminations since measures are countably additive. The vanishing of a semi-classical limit on the lift of a closed geodesic is provided by Corollary 10. It remains to consider vanishing for geodesic laminations. Consider an arc on $\mathbb{H}$ transverse to a measured geodesic lamination $\mathcal{G}$. Since the intersection is a closed perfect set there exist arbitrarily small subarcs non trivially intersecting $\mathcal{G}$. By Proposition 11 for small subarcs Proposition 9 can be applied to find that the set of intersecting leaves has measure zero for any semi-classical limit. The proof is complete.

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