# Families of Riemann Surfaces and Weil-Petersson Geometry

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# Families of Riemann Surfaces and Weil-Petersson Geometry

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# Preface

These written lectures are the companion to the NSF-CBMS Regional Research Conference in the Mathematical Sciences, organized July 20-24, 2009 at Central Connecticut State University, by Eran Makover and Jeffrey McGowan. My goal for the lectures is a generally self-contained course for graduate students and postgraduates. The topics run across current research areas. By plan the approach is didactic. Concepts are developed across multiple lectures. Opportunities are taken to introduce general concepts, to present recurring methods and to generally provide proofs. Guides to the research literature are included.

The study of Riemann surfaces continues to be an interface for algebra, analysis, geometry and topology. I hope that in part I am able to suggest the interaction to the audience and reader. The lectures are not intended as a proper research summary or history of the field. A collection of current and important topics are not included. Material is not always presented following the historical development of concepts. The references are not all inclusive but are intended only as a lead-in to the literature. Further readings are provided at the ends of chapters.

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Scott A. Wolpert
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